Utah State Mathematics Contest
Senior Exam
March 18, 2009

1. A metal cylinder is melted down and reshaped into a new cylinder. If the new diameter has been decreased by 20% without changing the volume, then the height has been increased by what percent?

   (a) 20 (b) 25 (c) 36 (d) 46.75 (e) 56.25

2. John has coins that may include pennies, nickels, dimes, or quarters. The mean value of the coins is 20 cents. If he were to add one quarter to his money, the new mean value would be 21 cents. How many quarters did he originally have?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

3. If the quadratic equation $x^2 - px - q = 0$ has two distinct real roots, then:

   (a) $p^2 - 4q \geq 0$ (b) $-p^2 - 4q > 0$ (c) $p^2 + 4q \geq 0$

   (d) $p^2 > 4q$ (e) $p^2 > -4q$

4. Evaluate the following expression:

$$\log_2 9 \cdot \log_3 25 \cdot \log_5 4$$

   (a) 2 (b) 6 (c) 8 (d) 9 (e) 30

5. How many positive integers less than 100,000 are both perfect squares and perfect cubes?

   (a) 6 (b) 7 (c) 8 (d) 9 (e) 10

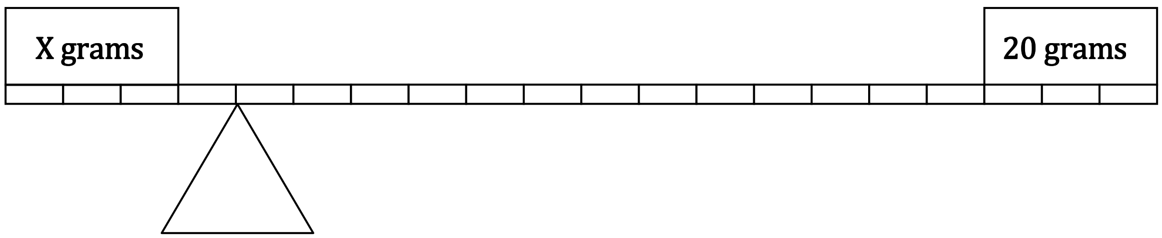
6. In the November 2008 general election, Utah voters turned out in strong numbers. Between Utah and Salt Lake counties, 67% of the 840 thousand registered voters made it to the election booth. Of registered voters, 65% of those in Salt Lake County voted, while 72% of those in Utah County voted. How many thousands of registered voters were in Utah County?

   (a) 230 (b) 240 (c) 250 (d) 260 (e) 270

7. When a line is drawn through a blank plane, it divides the plane into two sections. Further lines may divide the plane into more and more sections. Find the difference between the largest and smallest number of sections that may be created by drawing 9 distinct lines through a plane.

(a) 8 (b) 27 (c) 28 (d) 35 (e) 36

8. For what value of X would this system balance? (assume the lever has negligible mass)



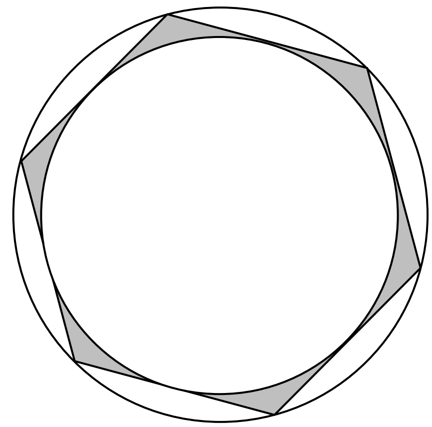
a) 148 (b) 260 (c) 116 (d) 140 (e) 80

9. How many distinct real solutions does the following equation have?

$$(4x^2 - 15x + 10)^{(4x^2 + 5x + 1)} = 1$$

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

10. The picture to the right illustrates a circle inscribed in a regular hexagon inscribed in another circle. What is the ratio of the combined areas of the shaded regions to the combined areas of the unshaded regions?



(a) $\frac{6\sqrt{3} - 3\pi}{7\pi - 6\sqrt{3}}$ (b) $\frac{4\pi - 6\sqrt{3}}{12\sqrt{3} - \pi}$ (c) $\frac{6\sqrt{3} - 3\pi}{6\pi - 4\sqrt{3}}$

(d) $\frac{6\sqrt{3} - 3\pi}{4\pi - 6\sqrt{3}}$ (e) $\frac{1}{12}$

11. If $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, then A^4 is equal to:

(a) $\begin{bmatrix} 16 & 32 \\ 0 & 16 \end{bmatrix}$ (b) $\begin{bmatrix} 16 & 16 \\ 0 & 16 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & 4 \\ 0 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} 8 & 16 \\ 0 & 8 \end{bmatrix}$ (e) $\begin{bmatrix} 16 & 1 \\ 0 & 16 \end{bmatrix}$

12. In the binomial expansion of $(x + y)^9$, what is the coefficient of the x^4y^5 term?

(a) 96 (b) 120 (c) 180 (d) 126 (e) 90

13. When written out in decimal form, how many zero-digits appear at the end of (2009!)?

(a) 200 (b) 222 (c) 401 (d) 500 (e) 2001

14. When Greg swims out from the beach, he is carried by the tide; it takes him 4 minutes to reach the nearest buoy. When he swims back in, he takes 16 minutes to swim against the tide. If Greg swims 100 yards per minute in still water, how many yards away is the buoy?

(a) 400 (b) 480 (c) 640 (d) 720 (e) 800

15. If $a > 1$, $\sin(x) > 0$, and $\log_a(\sin x) = c$, then $\log_a(\csc^2 x)$ is equal to:

(a) $\sqrt{1 - c^2}$ (b) $\frac{1}{c^2}$ (c) $-2c$ (d) c^{2a} (e) none of these

16. Find B if:

$$B = 1 + \cfrac{1 + \cfrac{1 + \cfrac{1+\cdots}{2+\cdots}}{2 + \cfrac{1+\cdots}{2+\cdots}}}{2 + \cfrac{1 + \cfrac{1+\cdots}{2+\cdots}}{2 + \cfrac{1+\cdots}{2+\cdots}}}$$

(a) $\frac{8}{5}$ (b) $\frac{\sqrt{5}+1}{2}$ (c) $\frac{1 \pm \sqrt{5}}{2}$ (d) $\frac{\sqrt{5}-1}{2}$ (e) ∞

17. What is the remainder when $(13x^{10} - 6x^7 + 12x^3 - 3x - 5)$ is divided by $(x + 1)$?

(a) 3 (b) 5 (c) 7 (d) 9 (e) 11

18. Let $A = \begin{bmatrix} x & 5 & 12 \\ 0 & 1 & x \\ 2 & 0 & x \end{bmatrix}$. What is the sum of all of the values of x for which A^{-1} doesn't exist?

(a) -4 (b) -6 (c) -8 (d) -9 (e) -10

19. Given that $7^y = 49^{x+6}$ and $32^x = 16^{y-9}$, then difference between x and y is:

(a) 8 (b) 9 (c) 12 (d) 15 (e) 16

20. Determine which of the following complex numbers is a solution to the equation:

$$x^6 + 1 = 0$$

(a) $\frac{\sqrt{3}}{2} - \frac{1}{2}i$ (b) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (c) $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ (d) $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ (e) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

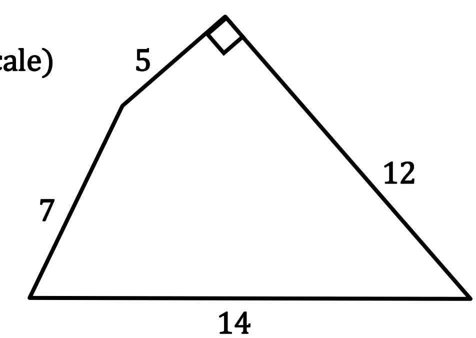
21. For some base, G, the number 359 would appear as 2009 in base 10. If $M = \frac{G-1}{3}$, what is the base M representation of 2009?

(a) 4027 (b) 2672 (c) 2009 (d) 3731 (e) 5200

22. Suppose that six brothers (Kojo, Largo, Milo, Nico, Otto, and Paulo) each roll a fair 6-sided die. At least three of them have rolled an even number. What is the probability that Largo, Paulo, and Milo have each rolled an even number?

(a) $\frac{7}{22}$ (b) $\frac{3}{32}$ (c) $\frac{1}{2}$ (d) $\frac{4}{17}$ (e) $\frac{4}{21}$

23. The quadrilateral pictured (not necessarily drawn to scale) contains one right angle, two acute angles, and one obtuse angle. Determine the area, in square units, of the given shape.



(a) $14\sqrt{30}$ (b) $30 + 2\sqrt{510}$ (c) $30 + \frac{1}{2}\sqrt{8281}$

(d) $30 + 14\sqrt{13}$ (e) $30 + 6\sqrt{38}$

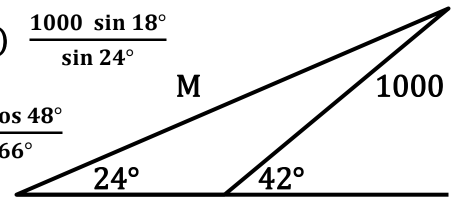
24. At 12:00 PM, the second, minute and hour hand on a clock all point to XII. Assuming continuous motion of all three hands, how many times will the second hand pass the minute hand in the time it takes for the minute hand to pass the hour hand three times?

(a) 192 (b) 193 (c) 194 (d) 195 (e) 196

25. What is the length of side M in the given triangle (not drawn to scale)?

(a) 1000 csc 24° (b) $\frac{1000}{\cos 18°}$ (c) $\frac{1000 \sin 18°}{\sin 24°}$

(d) $\frac{1000 \sin 42°}{\cos 24°}$ (e) $\frac{1000 \cos 48°}{\cos 66°}$



26. In an arithmetic sequence adding to 2009, the sum of the two least elements is one third of the sum of the three greatest. If the sequence has seven elements, determine the largest.

(a) 287 (b) $669\frac{2}{3}$ (c) 410 (d) 533 (e) $502\frac{1}{4}$

27. The following table includes the scoring totals for a six-match soccer tournament amongst four MLS teams (each team plays each other team once). In each match, a team gets two tournament points for a win and 0 points for a loss. If there is a tie, each team gets one tournament point.

TEAMS	Goals For	Goals Against	Tournament Points
DC United	2	1	4
LA Galaxy	2	5	1
New England Revolution	4	0	5
Real Salt Lake	1	3	2

NE vs DC ___ to ___
NE vs RSL ___ to ___
NE vs LA ___ to ___

DC vs RSL ___ to ___
DC vs LA ___ to ___
RSL vs LA ___ to ___

Determine the score in the match between DC United and Real Salt Lake (<u>DC to RSL</u>).

(a) 0 to 0 (b) 1 to 0 (c) 2 to 0 (d) 1 to 1 (e) 2 to 1

28. Mara and Lara are in a shooting competition. The object of the match is to be the first to hit the bullseye of a target at 100 feet. Each opponent has a 40% chance of hitting the bullseye on a given shot. If Lara graciously allows Mara to shoot first, what is the probability that Lara will win the competition?

(a) $\frac{2}{5}$ (b) $\frac{3}{10}$ (c) $\frac{3}{8}$ (d) $\frac{7}{20}$ (e) $\frac{13}{36}$

29. Find the sum of the following infinite series:

$$\frac{1}{4} + \frac{2}{16} + \frac{3}{64} + \frac{4}{256} + \dots$$

(a) $\frac{4}{7}$ (b) $\frac{3}{5}$ (c) $\frac{7}{15}$ (d) $\frac{1}{2}$ (e) $\frac{4}{9}$

30. What is the sum of the zeros of the function: $f(x) = x^4 - 6x^3 + 13x^2 - 14x + 6$?

(a) 0 (b) 4 (c) 6 (d) 9 (e) 14

31. There is a work crew that consists of 20% electricians, 30% plumbers, and 50% engineers. Amongst these people, 60% of the electricians have college degrees, 40% of the plumbers have degrees, and 80% of the engineers have degrees. If a randomly selected member of the crew has a college degree, what is the probability that this person is a plumber?

(a) $\frac{3}{25}$ (b) $\frac{3}{11}$ (c) $\frac{2}{9}$ (d) $\frac{3}{16}$ (e) $\frac{3}{10}$

32. Assume that you have 12 poles, each measuring a different whole number of meters ranging from 1 meter to 12 meters in length. How many distinct combinations of 3 poles can join end to end to form a triangle?

(a) 72 (b) 95 (c) 87 (d) 154 (e) 100

33. If x and y are real numbers such that $\frac{2y}{9x}$, $\frac{x}{2y}$, and $(12x - 8y)$ are all equivalent quantities, then $(9x + 6y)$ is equal to:

(a) -6 (b) -4 (c) $-\frac{9}{4}$ (d) $-\frac{3}{2}$ (e) 0

34. A researcher took a random sampling of 20 students and asked the amount of cash they had in their wallets. He found that the contents of the students' wallets had a mean of \$42, as well as a median of \$42. There were no perceived outliers in the group.
At this point, the researcher realized he had left a 21st student off of his calculations. The 21st student had \$80. After recalculating statistics for all 21 students, which of the following statements must be true?

(I) The mean increased
(II) The median increased
(III) The range increased
(IV) The mean is greater than the median

(a) I (b) I & IV (c) I, II & IV (d) I, III & IV (e) I, II, III & IV

35. In choosing a random real number, x, between 0 and 10, what is the probability that the following inequality holds true? (x is measured in radians)

$$\cot x \geq 1$$

(a) $\frac{40 - 9\pi}{40}$ (b) $\frac{\pi - 1}{10}$ (c) $\frac{4 - \pi}{4}$ (d) $\frac{40 - 6\pi}{40}$ (e) $\frac{3\pi}{40}$

36. Evaluate the limit: $\lim_{b\to\infty} \int_1^b \frac{x}{x^4 + 1}\, dx$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$ (e) $\frac{\pi}{10}$

37. There exists some positive real number, M, such that if you add five M's together, you get the same result as if you were to multiply five M's together. Which of the following is true about M?

(I) M^{15} is rational (II) M^{12} is rational (III) M^{30} is rational

(a) I only (b) II only (c) III only (d) II & III (e) I & III

38. What is the maximum value of B that would allow a real solution to the following equation? $2x^2 + 3y^2 - 18x - 18y + B = 0$

(a) 162 (b) $\frac{135}{4}$ (c) $\frac{135}{2}$ (d) $\frac{117}{4}$ (e) $\frac{117}{2}$

39. At the point specified by $\theta = \frac{\pi}{6}$, determine the slope of the line tangent to the curve:

$$r = 2 - \sin\theta$$

(a) $\frac{4-3\sqrt{3}}{11}$ (b) $\frac{-\sqrt{3}}{3}$ (c) $\frac{1-\sqrt{3}}{6}$ (d) $\frac{1-2\sqrt{3}}{3}$ (e) $\frac{-\sqrt{6}}{2}$

40. Five women are playing a card game in which they each have five cards. Each card is one of five different colors. All of the following statements are true:
(I) Any player who has a yellow card also has an orange card.
(II) Only if a player has a yellow and a red card does she have a blue card.
(III) A player has a green card if she does not have a yellow card.
(IV) A player does not have a blue card only if she does not have an orange card.
(V) Of blue, green, yellow and orange cards, each player has at least two colors.
Only one player currently has one card of each color. This woman is the only player holding a card that is:

(a) Red (b) Orange (c) Yellow (d) Green (e) Blue