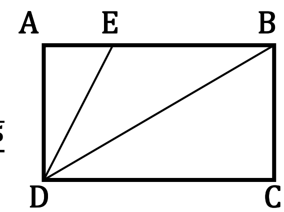
Utah State Mathematics Contest
Senior Exam
March 16, 2011

1. Making only vertical or horizontal moves between adjacent letters, in how many ways can you start from an *'M'* and move to the *'H'*, spelling the word *'MARCH'* ?

				M				
			M	A	**M**			
		M	A	R	A	**M**		
	M	A	R	C	R	A	**M**	
M	A	R	C	**H**	C	R	A	**M**
	M	A	R	C	R	A	**M**	
		M	A	R	A	**M**		
			M	A	**M**			
				M				

(a) 64 (b) 112 (c) 120 (d) 60 (e) 124

2. In rectangle ABCD, AD = 1, E is on AB, and DB and DE trisect $\angle$ADC. What is the perimeter of ΔBDE?



(a) $3+\frac{\sqrt{3}}{3}$ (b) $2+\frac{4\sqrt{3}}{3}$ (c) $2+2\sqrt{2}$ (d) $2+\frac{5\sqrt{3}}{3}$ (e) $\frac{3+\sqrt{5}}{2}$

3. If $A = Pe^{rt}$(where A, P, r, and t are all positive), what is the value of t when A is five times as much as P ?

(a) $5r$ (b) $\frac{1}{5}r$ (c) $\ln(5r)$ (d) $\frac{\ln(r)}{5}$ (e) $\frac{\ln(5)}{r}$

4. If the price of movie tickets goes up by p percent, what percent of decrease in ticket sales will result in no net change in income?

(a) $\frac{1}{1-p}$ (b) $\frac{p}{1+p}$ (c) $\frac{1-p}{p^2}$ (d) $\frac{1}{1+p}$ (e) $\frac{p}{1-p}$

5. The Utah Jazz and the LA Lakers play a five-game playoff series – the first team to win three games wins the series. Each team is equally likely to win each game, there are no ties, and the outcomes of the individual games are independent. If the Lakers were to win Game 2, but the Jazz were to win the entire series, what is the probability that the Lakers won Game 1?

(a) $\frac{1}{5}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$ (e) $\frac{2}{3}$

6. For how many positive integers, n , less than or equal to 24, is $n!$ evenly divisible by the sum $1+2+\cdots+n$? ($n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1$)

(a) 21 (b) 12 (c) 16 (d) 8 (e) 17

7. Let $\boldsymbol{a_1, a_2}, \dots$ be a sequence in which $\boldsymbol{a_1 = 7, a_2 = 10}$, and $\boldsymbol{a_n = \frac{a_{n-1}}{a_{n-2}}}$ for each integer $\boldsymbol{n \geq 3}$. What is the value of $\boldsymbol{a_{2011}}$?

(a) $\frac{10}{7}$ (b) 10 (c) $\frac{1}{7}$ (d) 7 (e) $\frac{7}{10}$

8. Two perpendicular lines both pass through the point $\boldsymbol{(6, 8)}$. These two lines have $\boldsymbol{y}$-intercepts that have a sum of zero. Determine the slope of the line with negative slope.

(a) $-\frac{1}{3}$ (b) $-\frac{1}{5}$ (c) $-\frac{1}{7}$ (d) $-\frac{1}{2}$ (e) $-\frac{3}{4}$

9. Consider the sets $\boldsymbol{\{0\}, \{1, 2\}, \{3, 4, 5\}, \{6, 7, 8, 9\}} \dots$ where the first set contains only the integer $\boldsymbol{0}$. For $\boldsymbol{n} > 1$, the $\boldsymbol{n^{th}}$ set contains $\boldsymbol{n}$ consecutive integers, and the first element in each set is the next integer after the last element in the preceding set. If $\boldsymbol{S_n}$ is the sum of the elements in the $\boldsymbol{n^{th}}$ set, find $\boldsymbol{S_{15}}$.

(a) 1120 (b) 570 (c) 1760 (d) 2240 (e) 1680

10. Thirty collectible action figures sell for an average of $20 each. Twenty more sell for an average of $30 each. What is the average price of all fifty collectibles?

(a) $23 (b) $24 (c) $25 (d) $26 (e) $27

11. The number of inches in the perimeter of an equilateral triangle equals the number of square inches in the area of a circle circumscribed about the triangle. What is the radius, in inches, of the circle?

(a) $\frac{3\sqrt{2}}{\pi}$ (b) $\frac{3\sqrt{3}}{\pi}$ (c) $\sqrt{3}$ (d) $\frac{6}{\pi}$ (e) $\sqrt{3}\pi$

12. Concrete sections on the new portions of I-15 are 80 feet long. As a car drives over the seams where sections meet, a brief thump is heard by the passengers. The speed of the car in miles per hour is approximately equal to the number of thumps heard in how many seconds?

(a) 70 (b) 45 (c) 55 (d) 85 (e) 40

13. The sum of the infinite series $\frac{4}{5} + \frac{5}{5^2} + \frac{4}{5^3} + \frac{5}{5^4} + \cdots$ is:

(a) $\frac{25}{16}$ (b) $\frac{15}{14}$ (c) $\frac{3}{2}$ (d) $\frac{5}{4}$ (e) $\frac{25}{24}$

14. Which if the following functions is defined as an odd function?

(a) $f(x) = \frac{x^3}{\sqrt{4x^2+1}}$ (b) $f(x) = x^3 + 5x + 9$ (c) $f(x) = \cos(7x)$
(d) $f(x) = \tan x + \sec x$ (e) $f(x) = 5x^2 + 3$

15. A traveler makes a 280 mile trip at an average speed of 7 miles per hour, but meanders home at an easy 4 miles per hour (yes, it's a long trip). The traveler's average speed for the entire trip is closest to:

(a) 5.0 mph (b) 5.7 mph (c) 5.3 mph (d) 5.5 mph (e) 5.1 mph

16. How many integer values of x can satisfy the following inequality?

$$\log(x-2) + \log(9-x) \leq 1$$

(a) 4 (b) 2 (c) 0 (d) 6 (e) infinitely many values

17. A barrel contains a selection of colored cubes, each of which is yellow, blue, or green. The number of green cubes is at least half of the number of blue cubes, and at most one third of the number of yellow cubes. The cubes which are blue or green number at least 34. The minimum number of yellow cubes is:

(a) 61 (b) 27 (c) 36 (d) 44 (e) 52

18. By adding the same particular constant to each of the numbers in the sequence $7, 15, 27$, a geometric progression is created. What is the common ratio between terms in the progression?

(a) $\frac{4}{3}$ (b) $\frac{8}{5}$ (c) $\frac{5}{3}$ (d) $\frac{3}{2}$ (e) 2

19. Let $M = i^n + i^{-n}$, where $i = \sqrt{-1}$, and n is any integer. How many different values can M possibly take?

(a) 2 (b) 3 (c) 4 (d) 6 (e) infinitely many

20. If the expressions $x + 3$ and $x - 3$ are reciprocals, what is the total of all possible values for x?

(a) $\mathbf{0}$ (b) $\sqrt{10}$ (c) $\sqrt{8}$ (d) $\mathbf{3}$ (e) $\sqrt{10} - \sqrt{8}$

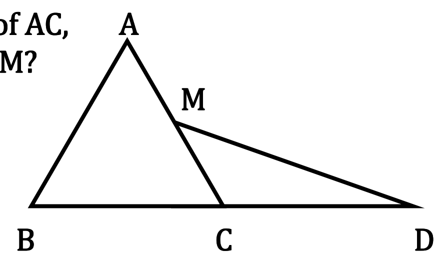
21. Let $f(x) = \frac{x}{1-x}$, where $x \neq 1$. Under such conditions, $f^{-1}(x)$ can be re-written as:

(a) $f\left(\frac{1}{x}\right)$ (b) $-f(x)$ (c) $-f(-x)$ (d) $f\left(-\frac{1}{x}\right)$ (e) $f(-x)$

22. There is a positive integer, B, such that $2B$ is a perfect square, $3B$ is a perfect cube, and $5B$ is a perfect fifth (an integer to the fifth power). Assume that B is the smallest such number. How many factors are in the prime factorization of B?

(a) 59 (b) 7 (c) 29 (d) 31 (e) 61

23. Equilateral ΔABC has side length 6. M is the midpoint of AC, and C is the midpoint of BD. What is the length of DM?



(a) $\frac{9\sqrt{2}}{2}$ (b) $3\sqrt{5}$ (c) $3\sqrt{6}$ (d) $3\sqrt{7}$ (e) $6\sqrt{2}$

24. Let x and y be two-digit integers such that y is obtained by reversing the digits of x. Suppose that the integers x and y also satisfy the equation $x^2 - y^2 = z^2$ for some positive integer z. What is the value of $x + y + z$?

(a) 144 (b) 116 (c) 88 (d) 154 (e) 112

25. If the radius of a circle is increased by 200%, then the area will be increased by:

(a) 200% (b) 800% (c) 300% (d) 900% (e) 400%

26. How many distinct real solutions are there to the following equation?

$$x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12 = 0$$

(a) 5 (b) 4 (c) 3 (d) 2 (e) 1

27. What is the largest number of acute angles that can be found in a convex decagon?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

28. Assuming that c is a positive integer, what is the largest number which necessarily divides the expression $c^4 - c^2$:

(a) 24 (b) 3 (c) 6 (d) 2 (e) 12

29. A right circular cylinder with diameter equal to its height is inscribed in a right circular cone (the bases and axes of the two objects would coincide). The cone has diameter 10 and altitude 12. What is the radius of the cylinder?

(a) $\frac{8}{3}$ (b) $\frac{30}{11}$ (c) 3 (d) $\frac{25}{8}$ (e) $\frac{7}{2}$

30. Imagine two circles, the larger with center M and radius m, the smaller with center N and radius n. Constructing line segment MN, which of the following choices cannot be true of the length of the line segment connecting the two centers?

(a) It is equal to $m - n$ (b) It is less than $m - n$ (c) It is equal to $m + n$

(d) It is less than $m + n$ (e) Any of these may be true

31. If the graph of the equation $9(x - h)^2 + 4(y - k)^2 = 36$ is tangent to both the x- and y-axes, then the sum of h and k may be:

(a) 6 (b) 4 (c) -1 (d) 3 (e) -9

32. The areas (in square inches) of the front, top, and side of a rectangular box are 3, 10, and 12, respectively. Which of the following choices is the closest to the measure of the volume of the box in cubic inches?

(a) 22 (b) 27 (c) 19 (d) 16 (e) 25

33. In the equation $C = \frac{An}{B+nZ}$, let A, B, and Z be positive constants, and let n be a positive integer. As n increases, what happens to C?

(a) it increases (b) it decreases (c) it remains constant

(d) in increases, then decreases (e) it decreases, then increases

34. When the mean, median, and mode of the list $\{10, 2, 5, 2, 4, 2, x\}$ are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

(a) 6 (b) 20 (c) 9 (d) 3 (e) 17

35. The expression $\sqrt{5+2\sqrt{6}} - \sqrt{5-2\sqrt{6}}$ is equivalent to which of the following?

(a) $2\sqrt{6}$ (b) $2\sqrt{2}$ (c) $3\sqrt{5}$ (d) $3\sqrt{3}$ (e) $5\sqrt{2}$

36. Let m, n, and p be real numbers such that $8m + 9n - p = 12$ and $9m - 8n + 12p = 1$. Find the sum of $m^2 + n^2 - p^2$.

(a) 12 (b) 0 (c) 8 (d) 1 (e) 9

37. If $\log_A B = \log_B A$, where $A > B > 0$, then the product AB is:

(a) 1 (b) 2 (c) 10 (d) $\frac{1}{2}$ (e) not a real number

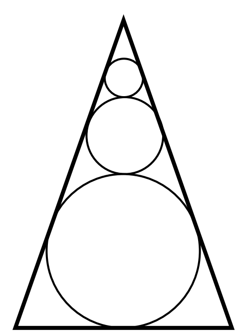
38. If $y = mx + b$ is the slant asymptote of $f(x) = \frac{5x^2-11}{x-3}$, then which is equal to $m^2 + b$?

(a) 10 (b) 29 (c) 40 (d) 59 (e) 74

39. Let P be the product of the numbers 3162011, 6102011, and 7122011. The value of $\log P$ is closest to which of the following?

(a) 19 (b) 22 (c) 20 (d) 23 (e) 21

40. Ben draws an image of Frosty the Snowman, comprised of three circles of radii 1, 2, and 4 inches. He then draws a triangular shade over Frosty (to keep out the sun) which runs tangent to Frosty's circular parts (which are, of course, tangent to each other). What is the area enclosed by the triangular shade?



(a) $\frac{128\sqrt{2}}{3}$ (b) $\frac{128\sqrt{3}}{3}$ (c) $32\sqrt{5}$ (d) $\frac{128\sqrt{3}}{3}$ (e) $64\sqrt{2}$