Utah State Mathematics Contest
Senior Exam
March 14, 2012

**1. If $n = 2^{20}$, what is the sum of all integers $0 < d \le n$ that divide $n$?**

(a) $2^{22}$ (b) $2^{22}-1$ (c) $2^{21}-1$ (d) $2^{22}+1$ (e) $2^{21}+1$

**2. The number of vertices of an ordinary polyhedron with 13 faces and 19 edges is**

(a) 7 (b) 6 (c) 10 (d) 8 (e) 12

**3. A cube is such that the length of its longest diagonal in centimeters is the same as its volume in cubic centimeters. What is the length in centimeters of each side of the cube?**

(a) $\sqrt[4]{3}$ cm (b) $\sqrt[4]{27}$ cm (c) $\sqrt[4]{8}$ cm (d) $\sqrt[4]{6}$ cm (e) $\sqrt[4]{9}$ cm

**4. The first fifteen digits of pi are 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9. How many distinct ten digit numbers can be formed with these digits.**

(a) $\frac{15!}{5}$ (b) $\frac{15!}{5!}$ (c) $\frac{15!}{36}$ (d) $\frac{15!}{120 \cdot 144}$ (e) $\frac{15!}{180}$

**5. For $f(x) = \frac{2x}{1+2x}$, find $f(f(f(x)))$.**

(a) $\frac{2x^3}{1+2x^3}$ (b) $\frac{8x}{1+14x}$ (c) $\frac{2x}{1+8x}$ (d) $\frac{8x^3}{1+16x^3}$ (e) $\frac{2x}{1+16x}$

**6. If the unspecified coefficients $a_1, a_2, a, a_4, a_5, a_6, a_7, a_8,$ and $a_9$ are real numbers, what is the maximum number of rational solutions the following polynomial equation could have?**

$$2x^{10} + a_9x^9 + a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x - 5 = 0$$

(a) 8 (b) 10 (c) 6 (d) 2 (e) 4

**7. In a circle of radius $\frac{4}{\pi}$, find the area between an arc of length 2 and its chord.**

(a) $\frac{4}{\pi}$ (b) $\frac{8}{\pi}$ (c) $\frac{2}{\pi}-\frac{4}{\pi^2}$ (d) $\frac{2}{\pi}$ (e) $\frac{4}{\pi}\left[1-\frac{2}{\pi}\right]$

**8. A train, traveling at a constant speed, takes 20 seconds from the time it enters a tunnel that is 300 meters long until it emerges from the tunnel. A bat sleeping on the ceiling of the tunnel is directly above the train for 5 seconds. How long is the train?**

(a) 250 m (b) 100 m (c) 150 m (d) 200 m (e) 300 m

**9. If $f(e^x) = \sqrt{x}$, then $f^{-1}(x)$ is**

(a) $(\ln x)^2$ (b) $e^{\sqrt{x}}$ (c) $e^{x^2}$ (d) $\sqrt{\ln x}$ (e) $2\ln x$

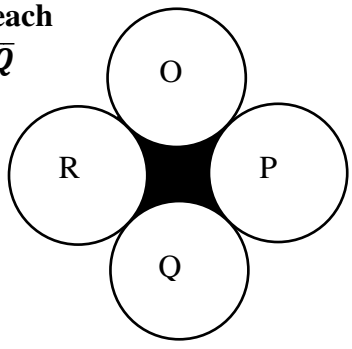
**10. An archery target has two scoring areas: one worth 5 points and another worth 7 points. What is the largest score impossible to obtain?**

(a) 18 (b) 16 (c) 22 (d) 23 (e) 19

**11. The number 144 can be expressed as the difference of perfect squares $x^2 - y^2$ in four different ways. Find the largest value of such $x$.**

(a) 35 (b) 16 (c) 24 (d) 20 (e) 37

**12. In the figure, circles O and Q are tangent to circles R and P and each circle has radius 1. What is the area of the shaded region if $\overline{OP} \perp \overline{PQ}$ and $\overline{OP} \perp \overline{OR}$?**




(a) $\frac{\pi}{4}$ (b) 1 (c) $\pi$ (d) $4 - \pi$ (e) 4

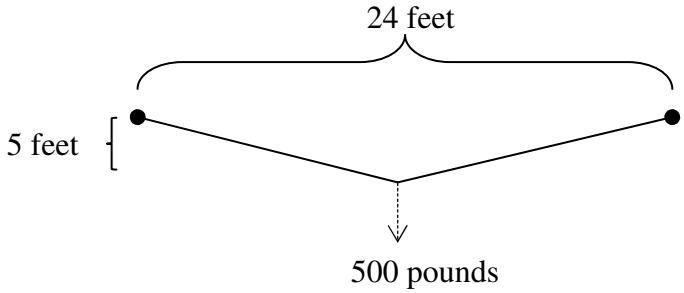
**13. Suppose you have a circular pizza. Into how many pieces (not necessarily the same size) can the pizza be cut using 5 straight lines?**

(a) 15 (b) 11 (c) 16 (d) 10 (e) 12

**14. Twenty-four people are gathered in a room. Beginning at 11 am, everyone shakes hands with everyone else. Each handshake takes 3 seconds with 3 seconds between handshakes. If 12 pairs of people are shaking hands simultaneously, at what time is the handshaking completed?**

(a) 11:03:10 (b) 11:02:27 (c) 11:03:15 (d) 11:03:07 (e) 11:02:15

**15. A 500 pound object is hung at the center of a 26 foot cable anchored to a wall at each end as shown in the following diagram. How much force is exerted on each anchor?**




(a) 650 lbs (b) 600 lbs (c) 450 lbs (d) 500 lbs (e) 250 lbs

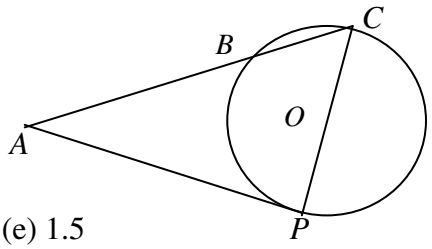
**16. A circle with center $(h, k)$ contains the points (–2, 10), (–9, –7), and (8, –14). The circumference of the circle is**

(a) $10\pi$ (b) $26\pi$ (c) $24\pi$ (d) $13\pi$ (e) $12\pi$

**17. From a group of 15 mathematics students, 10 were randomly selected to be on a state mathematics team. Let $P$ represent the probability that 4 of the 5 top students are included in the selection. Which of the following statements is true?**

(a) $\frac{1}{5} \le P \le \frac{2}{5}$ (b) $\frac{3}{5} \le P \le \frac{4}{5}$ (c) $\frac{2}{5} \le P \le \frac{3}{5}$ (d) $0 \le P \le \frac{1}{5}$ (e) $\frac{4}{5} \le P \le 1$

**18. In the circle shown with center *O* and radius 2, *AP* has length 3 and is tangent to the circle at *P*. If *CP* is the diameter of the circle, what is the length of *BC*?**




(a) 5 (b) 3.2 (c) 3 (d) 2 (e) 1.5

**19. Find the equation of the set of all points $(x, y)$ such that the sum of those distances from (0, 1) and (1, 0) is 2.**

(a) $x^2 + xy + y^2 - 2x - 2y + 2 = 0$ (b) $3x^2 - 2xy + 3y^2 - 4x + 4y - 2 = 0$

(c) $4x^2 - 2xy + 4y^2 - 2x - 2y = 0$ (d) $x^2 - 2xy + y^2 + 2x - 2y + 4 = 0$

(e) $3x^2 + 2xy + 3y^2 - 4x - 4y = 0$

**20. A witness to a robbery tells police that the license of the car contained three digits, the first of which was 9, followed by three letters, the last of which was A. The witness cannot remember the second and third digit nor the first and third letter, but is certain that all the numbers were different and that the first two letters were the same but different from the last letter. How many possible license plates match the description?**

(a) 1872 (b) 2600 (c) 2106 (d) 729 (e) 1800

**21. If $2f(x) + f(1 - x) = x^2$ for all $x$, find $f(x)$.**

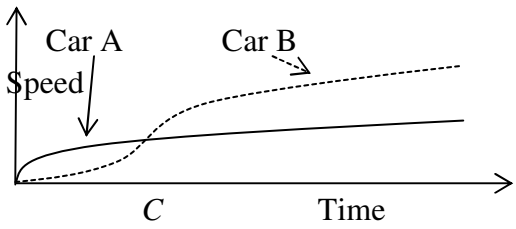
(a) $f(x) = \frac{1}{5}(x^2 + 2x - 1)$ (b) $f(x) = \frac{1}{3}(x^2 + 2x - 1)$ (c) $f(x) = \frac{1}{3}(x - 1)^2$

(d) $f(x) = -\frac{1}{2}(x - 1)^2$ (e) $f(x) = \frac{1}{2}(x - 1)^2$

**22. If $2^x = 15$ and $15^y = 32$, then $xy$ =**

(a) 5 (b) 7 (c) 9 (d) 11 (e) 13

**23. Car A (solid graph) and Car B (dotted graph) race from a standing start and travel in a straight path, initially side-by-side. A graph of the speed of the two cars is shown. At time *C*, the graphs intersect. Which description correctly describes the relative position and behavior of the two cars at time *C*?**




(a) Car A and Car B are even but Car B is moving faster

(b) Car A and Car B are even and moving at the same speed

(c) Car A is ahead of Car B but Car B is moving faster

(d) Car A is ahead of Car B and both cars are moving at the same speed

(e) Car A is behind Car B but Car A is moving faster than Car B

**24. The equation $r = 2\sin\theta - \cos\theta$ in rectangular coordinates is given by**

(a) $x^2 + y^2 + 2x - y = 0$ (b) $x^2 - x + 2y = 0$ (c) $x^2 + y^2 + x - 2y = 0$

(d) $x^2 - y^2 - x + 2y = 0$ (e) $y^2 - x^2 - x + 2y = 0$

**25. Let R be the region bounded by $y = x - 1$, $x = 1$, and $y = 3 - x$. Find the maximum value of $f(x, y) = -2x + 3y$ on the region R.**

(a) –2 (b) 4 (c) 2 (d) 1 (e) –1

**26. Find the value of the sum $\frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$.**

(a) $\frac{7}{3}$ (b) 3 (c) $\frac{11}{9}$ (d) $\frac{4}{3}$ (e) $\infty$

**27. In a computer program, separate loops with distinct indices produce *M* and *N* operations, respectively. If these reside internally in a loop with an independent index producing *P* operations, find the total number of operations represented by the three loops.**

(a) $P^{M+N}$ (b) $(M+N)^P$ (c) $P(M+N)$ (d) $P^M + P^N$ (e) $M^P + N^P$

**28. Assuming that a person selects an answer to each of the first ten questions on this examination at random and that the selections are independent, what is the probability that he/she will correctly guess exactly five answers?**

(a) $\frac{(67)4^6}{5^{10}}$ (b) $\frac{(65)4^6}{5^{10}}$ (c) $\frac{4^9}{5^{10}}$ (d) $\frac{(61)4^6}{5^{10}}$ (e) $\frac{(63)4^6}{5^{10}}$

**29. Given the recursion $x_{n+2} + 6x_{n+1} + 9x_n = 0$ ($n$ = 0, 1, 2, …) with $x_0 = 1, x_1 = 0$, then the value of $x_5$ equals**

(a) 972 (b) −834 (c) 1944 (d) −243 (e) 774

**30. You are traveling through an enchanted land when you come to a bridge that leads to a region of mathematical mysteries which is guarded by a troll. Beside the troll, there is a fountain of water and two empty jugs both sitting on a scale. The jugs have capacities 3 gallons and 5 gallons. The troll says that in order to cross the bridge, both jugs must be returned to the scale and the scale must read that 4 gallons of water occupy it. However, you must accomplish this with the fewest number of pours or else you will be cast into a river of fire.**

**To measure 4 gallons of water you must take water from the fountain using the jugs and you can pour the contents of one jug into the other and into the fountain. What is the minimum number of pours required by the troll to measure 4 gallons? (Filling a jug from the fountain is considered a pour and emptying all or part of a jug is also a pour.)**

(a) 4 (b) 6 (c) 7 (d) 9 (e) 5