Utah State Mathematics Contest
Senior Exam Solutions
March 14, 2012

1. If $n = 2^{20}$, what is the sum of all integers $0 < d \le n$ that divide n?
Solution: (c) Looking at the first few powers of 2 and the sum of the divisors, we find

2^0	2^1	2^2	2^3	2^4
1	1 + 2 = 3	1 + 2 + 4 = 7	1 + 2 + 4 + 8 = 15	1 + 2 + 4 + 8 + 16 = 31

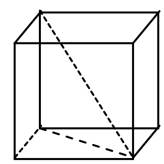
In general, if S_n denotes the sum of the divisors of 2^n, then $S_n = 2^{n+1} - 1$. In this case, the sum is $2^{21} - 1$.

2. The number of vertices of an ordinary polyhedron with 13 faces and 19 edges is
Solution: (d) Euler's formula $F - E + V = 2$ so $V = 8$.

3. A cube is such that the length of its longest diagonal in centimeters is the same as its volume in cubic centimeters. What is the length in centimeters of each side of the cube?
Solution: (a) Let L denote the length of the (4) longest diagonals and let x denote the side length of the cube. Then

$$L = x^3$$

but we also have that $L = \sqrt{3}\,x$. Equating expressions for L, we obtain that $x = \sqrt[4]{3}$ cm.



4. The first fifteen digits of pi are 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9. How many distinct ten digit numbers can be formed with these digits.
This problem was to count the permutations of the first *fifteen* (not ten) digits of pi. It has been thrown out.

5. For $f(x) = \dfrac{2x}{1+2x}$, find $f(f(f(x)))$.
Solution: (b) Initially, we have

$$f(f(x)) = \frac{2\left(\frac{2x}{1+2x}\right)}{1+2\left(\frac{2x}{1+2x}\right)} \cdot \frac{1+2x}{1+2x} = \frac{4x}{1+2x+4x} = \frac{4x}{1+6x}$$

Composing a second time gives

$$f(f(f(x))) = \frac{2\left(\dfrac{4x}{1+6x}\right)}{1+2\left(\dfrac{4x}{1+6x}\right)} \cdot \frac{1+6x}{1+6x}$$
$$= \frac{8x}{1+6x+8x}$$
$$= \frac{8x}{1+14x}$$

6. If the unspecified coefficients $a_1, a_2, a, a_4, a_5, a_6, a_7, a_8,$ and a_9 are real numbers, what is the maximum number of rational solutions the following polynomial equation could have?

$$\mathbf{2x^{10} + a_9x^9 + a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x - 5 = 0}$$

Solution: (a or b) As stated, there could be as many as 10 rational solutions if the unknown coefficients are rational numbers. For an illustration, consider the factored form

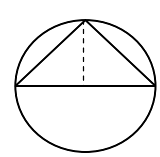
$$(2x+1)(x-5)(x-r_1)\left(x-\frac{1}{r_1}\right)(x-r_2)\left(x-\frac{1}{r_2}\right)(x-r_3)\left(x-\frac{1}{r_3}\right)(x-r_4)\left(x-\frac{1}{r_4}\right) = 0$$

which is a tenth degree polynomial having leading coefficient 2 and constant term –5 allowing for ten rational solutions.

The Rational Root Theorem requires integer coefficients in specifying a more restricted structure of the rational solutions. In that case, the rational solutions have the form $\pm\frac{p}{q}$ where p divides the constant term –5 and q divides the leading coefficient 2. The possible rational solutions in this case would be $\pm 1, \pm 5, \pm\frac{1}{2}, \pm\frac{5}{2}$. We will accept both answers.

7. In a circle of radius $\frac{4}{\pi}$, find the area between an arc of length 2 and its chord.

Solution: (e) The circumference of this circle is $2\pi\left(\frac{4}{\pi}\right) = 8$. Therefore, a chord of length 2 spans a quarter of the circle. Two such chords span a half circle and form the legs of a right triangle having a diametrical base and radial altitude. The area we seek is given by



$$\frac{1}{2}(\text{Area of Half Circle} - \text{Area of Triangle}) = \frac{1}{2}\left[\frac{\pi}{2}\left(\frac{4}{\pi}\right)^2 - \frac{1}{2}\left(\frac{8}{\pi}\right)\left(\frac{4}{\pi}\right)\right]$$
$$= \frac{4}{\pi} - \frac{8}{\pi^2}$$

8. A train, traveling at a constant speed, takes 20 seconds from the time it enters a tunnel that is 300 meters long until it emerges from the tunnel. A bat sleeping on the ceiling of the tunnel is directly above the train for 5 seconds. How long is the train?
Solution: (b) The train travels 300 meters plus its length in 20 seconds and travels its length in 5 seconds. Therefore, it travels four times its length in 20 seconds: $300 + L = 4L$ or $3L = 300$ m. The train is 100 m long.

9. If $f(e^x) = \sqrt{x}$, then $f^{-1}(x)$ is
Solution: (c) If $f(e^x) = \sqrt{x}$, then $f(x) = \sqrt{\ln x}$ for $x \geq 1$. The inverse of f is found by solving

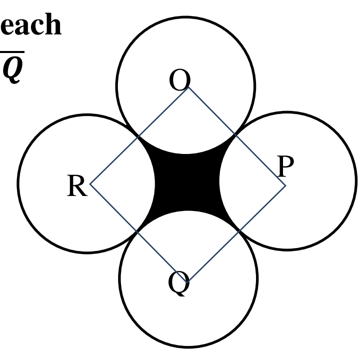
$$x = \sqrt{\ln y}$$

for y where $x^2 = \ln y$ so $f^{-1}(x) = e^{x^2}$.

10. An archery target has two scoring areas: one worth 5 points and another worth 7 points. What is the largest score impossible to obtain?
Solution: (d) First, we note that 5 and 7 are relatively prime. Clearly, if both numbers were even, no odd score would be possible. In listing the possible scores, once there are 5 consecutive possible scores, all scores after that are possible. Certainly, all multiples of 5 and 7 are possible. The possible scores are
5, 7, 10, 12 (5+7), 14, 15, 17 (10+7), 19 (14+5), 20, 21, 22 (15+7), **24** (19+5)**,** **25**, **26** (19+7)**,** **27** (20+7)**,** **28,** (that is five in a row…), 24+5=29, 30, 26+5=31, …
Thus, 23 is the largest score impossible to obtain. (Because 5 and 7 are relatively prime, this result can be obtained by $5 \cdot 7 - 5 - 7 = 23$.)

11. The number 144 can be expressed as the difference of perfect squares $x^2 - y^2$ in four different ways. Find the largest value of such x.
Solution: (e) Using the fact that $x^2 - y^2 = (x - y)(x + y)$, we can factor 144 using $x - y$ and $x + y$. Furthermore, the sum of these factors is $2x$, so the sum of the two factors must be even to produce a difference of squares. We consider the list of all factor pairs of 144:
1 and 144 (sum: 145), **2 and 72 (sum: 74; $x = 37$)** 3 and 48 (sum: 51), 4 and 36 (sum: 40; $x = 20$), 6 and 24 (sum: 30; $x = 15$), 8 and 18 (sum: 26; $x = 13$). (Yes, if you include $12^2 + 0^2$, there would be 5 ways…)

12. In the figure, circles O and Q are tangent to circles R and P and each circle has radius 1. What is the area of the shaded region if $\overline{OP} \perp \overline{PQ}$ and $\overline{OP} \perp \overline{OR}$?

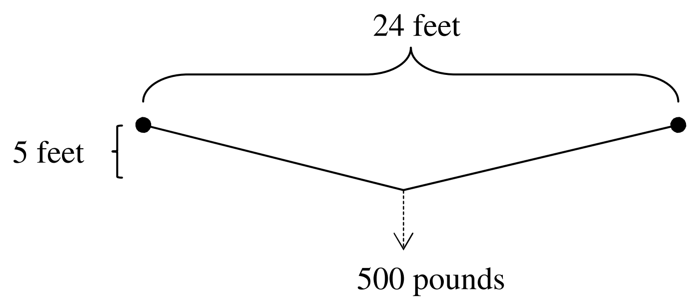


Solution: (d) The centers of the circles are vertices of a square with sides of length 2 which intersects each circle in a quarter circle. Thus, the shaded area is $4 - \pi$.

13. Suppose you have a circular pizza. Into how many pieces (not necessarily the same size) can the pizza be cut using 5 straight lines?
This problem was to determine the *maximum* number of pieces possible and was thrown out. All of the given number of pieces are *possible* but 16 is the maximum number possible.

14. Twenty-four people are gathered in a room. Beginning at 11 am, everyone shakes hands with everyone else. Each handshake takes 3 seconds with 3 seconds between handshakes. If 12 pairs of people are shaking hands simultaneously, at what time is the handshaking completed?
Solution: (e) Every person must shake hands 23 times. Since everyone is shaking hands simultaneously, there are 23 handshake periods. Including the 3 seconds between handshaking, there are $23 \cdot 6 - 3$ seconds omitting the last 3 second transition because the handshaking has been completed. From start to finish, there is 135 seconds required or 2 minutes and 15 seconds.

15. A 500 pound object is hung at the center of a 26 foot cable anchored to a wall at each end as shown in the following diagram. How much force is exerted on each anchor?



Solution: (a) We note that the triangle formed by force along the cable with the horizontal and vertical forces is proportional to the 5:12:13 right triangle formed by the cable. Each anchor supports exactly half of the 500 pound weight by its vertical component. Therefore, the weight induces a force F along the cable itself that satisfies the proportion

$$\frac{5}{13} = \frac{250}{F}$$

Solving the proportion gives $F = 13 \cdot 50 = 650$ lbs.

16. A circle with center (h, k) contains the points (–2, 10), (–9, –7), and (8, –14). The circumference of the circle is
Solution: (b) We establish two equations in h and k by equating the distance from the center to two points.

From (–2, 10) to (–9, –7)

$$(h+2)^2 + (k-10)^2 = (h+9)^2 + (k+7)^2$$

From (–2, 10) to (8, –14)

$$(h+2)^2 + (k-10)^2 = (h-8)^2 + (k+14)^2$$

Expanding, the quadratic terms add out leaving

$$14h + 34k = -26 \qquad\qquad 20h - 48k = 156$$

from which we obtain $h = 3$ and $k = -2$. The radius of the circle is $\sqrt{(3+2)^2 + (-2-10)^2} = 13$.
The circumference is 26π.

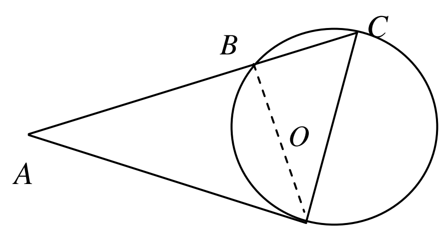
17. From a group of 15 mathematics students, 10 were randomly selected to be on a state mathematics team. Let *P* represent the probability that 4 of the 5 top students are included in the selection. Which of the following statements is true?

Solution: (a) There are 5 ways to select 4 students from 5. This is independent of selecting the last 5 team members from the remaining 10. Hence, the probability P is

$$P = \frac{5 \cdot \binom{10}{6}}{\binom{10}{5}} = 5 \cdot \frac{10!}{6!4!} \cdot \frac{10!5!}{15!} = \frac{50}{143}$$

Furthermore, $.2 < P < .4$.

18. In the circle shown with center *O* and radius 2, *AP* has length 3 and is tangent to the circle at *P*. If *CP* is the diameter of the circle, what is the length of *BC*?



Solution: (b) We note that $\overline{AP} \perp \overline{CP}$ so triangle APC is a right triangle. Therefore, the length of $\overline{AC}$ is 5. Triangle PBC is also a right triangle because segment $\overline{CP}$ is a diameter. Therefore, triangle PBC is similar to triangle APC. Thus, the ratio of $\overline{BC} : \overline{CP}$ is 4:5. Because $\overline{CP} = 4$, solving for $\overline{BC}$ gives $\overline{BC} = 3.2$.

19. Find the equation of the set of all points (*x*, *y*) such that the sum of those distances from (0, 1) and (1, 0) is 2.

Solution: (e) We have

$$\sqrt{x^2 + (y-1)^2} + \sqrt{(x-1)^2 + y^2} = 2$$

or

$$\left(\sqrt{x^2 + (y-1)^2}\right)^2 = \left(2 - \sqrt{(x-1)^2 + y^2}\right)^2$$

which produces

$$x^2 + (y-1)^2 = 4 - 4\sqrt{(x-1)^2 + y^2} + (x-1)^2 + y^2$$

Isolating the remaining radical and squaring both sides yields

$$(x-y-2)^2 = \left(-2\sqrt{(x-1)^2+y^2}\right)^2$$

Expanding and collecting like terms results in

$$3x^2 + 2xy + 3y^2 - 4x - 4y = 0$$

20. A witness to a robbery tells police that the license of the car contained three digits, the first of which was 9, followed by three letters, the last of which was A. The witness cannot remember the second and third digit nor the first and second letter, but is certain that all the numbers were different and that the first two letters were the same but different from the last letter. How many possible license plates match the description?

Solution: (e) Given that the last letter was 'A', the witness could not remember the first and *second* letters. There are 9 possibilities for the second digit, 8 possibilities for the third digit, and 25 possibilities for the double letters: $9\cdot8\cdot25 = 1800$.

21. If $2f(x) + f(1-x) = x^2$ for all x, find $f(x)$.

Solution: (b) Solve for $f(x)$ to get

$$f(x) = \tfrac{1}{2}x^2 - \tfrac{1}{2}f(1-x). \qquad (*)$$

Now consider $f(1-x)$ noting that $1-(1-x) = x$:

$$f(1-x) = \tfrac{1}{2}(1-x)^2 - \tfrac{1}{2}f(x)$$

Substituting this expression into equation (*) produces

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 - \frac{1}{2}\left[\frac{1}{2}(1-x)^2 - \frac{1}{2}f(x)\right] \\ &= \frac{1}{2}x^2 - \frac{1}{4}(1-2x+x^2) + \frac{1}{4}f(x) \\ &= \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{4} + \frac{1}{4}f(x) \end{aligned}$$

Solving for $f(x)$ yields

$$f(x) = \tfrac{1}{3}x^2 + \tfrac{2}{3}x - \tfrac{1}{3}.$$

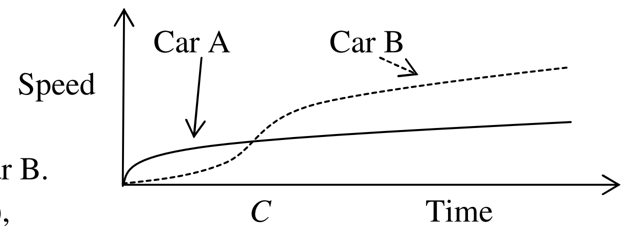
22. If $2^x = 15$ and $15^y = 32$, then $xy =$

Solution: (a) We have

$$\begin{aligned} 32 &= 15^y \\ &= (2^x)^y \\ &= 2^{xy} \end{aligned}$$

So $xy = 5$.

23. Car A (solid graph) and Car B (dotted graph) race from a standing start and travel in a straight path, initially side-by-side. A graph of the speed of the two cars is shown. At time C, the graphs intersect. Which description correctly describes the relative position and behavior of the two cars?



Solution: (a, c, d) We interpret the graphs as follows: Car A has greater speed initially so it moves ahead of Car B. By the time the two cars have the same speed (at time *C*), Car A is ahead of Car B and they have the same speed. We will accept answers (a) and (c) as feasible.

24. The equation $r = 2\sin\theta - \cos\theta$ in rectangular coordinates is given by

Solution: (c) Using polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. By multiplying both sides of the equation by *r*, we obtain

$$r^2 = 2r\cos\theta - r\sin\theta$$

which equates to the rectangular form

$$x^2 + y^2 = 2y - x$$

or

$$x^2 + y^2 + x - 2y = 0$$

25. Let R be the region bounded by $y = x - 1$, $x = 1$, and $y = 3 - x$. Find the maximum value of $f(x, y) = -2x + 3y$ on the region R.

Solution: (b) The graph of the region is the triangle with vertices (1, 0), (1, 2), and (2, 1). Because *x* is positive throughout this region, we seek a point with the smallest value of *x* and the largest value of *y*. This occurs at (1, 2) with $f(1, 2) = 4$.

26. Find the value of the sum $\frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$.

Solution: (d) The series is a convergent geometric series with a constant ratio between terms of $r = \frac{2}{3}$. The sum of the series is given by

$$S = \frac{\frac{4}{9}}{1 - \frac{2}{3}} = \frac{4}{9} \cdot \frac{3}{1} = \frac{4}{3}$$

27. In a computer program, separate loops with distinct indices produce *M* and *N* operations, respectively. If these reside internally in a loop with an independent index producing *P* operations, find the total number of operations represented by the three loops.

Solution: (c) Each instance of the outer loop executes the inner loops resulting in $M + N$ operations. Therefore, *P* executions of the outer loop produces $P(M + N)$ operations.

28. Assuming that a person selects an answer to each of the first ten questions on this examination at random and that the selections are independent, what is the probability that he/she will correctly guess exactly five answers?

Solution: (e) The probability of a correct answer is $\frac{1}{5}$ while the probability of an incorrect answer is $\frac{4}{5}$. The number of ways to obtain a ten-tuple with 5 correct answers is $\binom{10}{5} = \frac{10!}{5!5!} = 252.$ Each of these has probability $\left(\frac{1}{5}\right)^5 \left(\frac{4}{5}\right)^5 = \frac{4^5}{5^{10}}$ so the probability of guessing exactly five correct answers on the first ten questions is

$$252 \cdot \frac{4^5}{5^{10}} = \frac{63 \cdot 4^6}{5^{10}}$$

29. Given the recursion $x_{n+2} + 6x_{n+1} + 9x_n = 0$ (n = 0, 1, 2, ...) with $x_0 = 1, x_1 = 0$, then the value of x_5 equals

Solution: (a) Taking a direct approach, we have $x_{k+2} = -6x_{k+1} - 9x_k$, so

$x_2 = -6(0) - 9(1) = -9$, $x_3 = -6(-9) - 9(0) = 54$, $x_4 = -6(54) - 9(-9) = -243$

$x_5 = -6(-243) - 9(54) = 972$

30. You are traveling through an enchanted land when you come to a bridge that leads to a region of mathematical mysteries which is guarded by a troll. Beside the troll, there is a fountain of water and two empty jugs both sitting on a scale. The jugs have capacities 3 gallons and 5 gallons. The troll says that in order to cross the bridge, both jugs must be returned to the scale and the scale must read that 4 gallons of water occupy it. However, you must accomplish this with the fewest number of pours or else you will be cast into a river of fire.

To measure 4 gallons of water you must take water from the fountain using the jugs and you can pour the contents of one jug into the other and into the fountain. What is the minimum number of pours required by the troll to measure 4 gallons? (Filling a jug from the fountain is considered a pour and emptying all or part of a jug is also a pour.)

Solution: (c) Fill the 5 gallon jug (1st pour) from the fountain. Fill the 3 gallon jug (2nd pour) from the 5 gallon jug leaving 2 gallons in the 5 gallon jug. Empty the 3 gallon jug into the fountain (3rd). Pour the remaining 2 gallons (4th) from the 5 gallon jug into the 3 gallon jug. Refill the 5 gallon jug (5th) and use this to fill (6th) the 3 gallon jug leaving 4 gallons in the 5 gallon jug. Empty the 3 gallon jug in the fountain (7th pour).