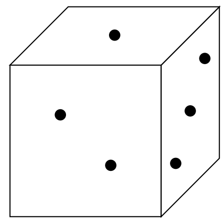
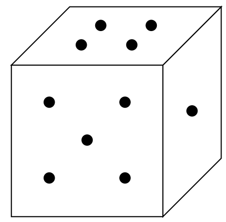
State Math Contest – Senior Exam

SOLUTIONS

1. The following pictures show two views of a non standard die (however the numbers 1 - 6 are represented on the die). How many dots are on the bottom face of figure 2?



1



2

a) 1 b) 2 c) 3

d) 4 e) 5

Solution:

Correct answer: 2 (**b**)
Rotate picture 1 90 degrees clockwise (so the 1 is on the right as it is in picture 2). Then imagine rotating the resulting cube 90 degrees forward so the 2 and 3 are now on the bottom and rear face respectively. It's the only way to make them both disappear so we can see the 4 and 5 as we do in image 2.

2. Two bicyclists, Annie and Bonnie, are 30 miles apart on a steep road. Annie and Bonnie travel at a constant speed and start riding towards each other at the same time. Annie travels downhill and goes twice as fast as Bonnie. They expect to meet in one hour, but Annie stops for a flat tire after 30 minutes and she is unable to continue. How many minutes should Annie expect to wait for Bonnie if Bonnie continues at the same speed?

a) 45 b) 60 c) 75

d) 90 e) 105

Solution:

Correct answer: 90 (**d**)
If they expect to meet in 1 hour, then together they must travel the total distance of 30 miles in one hour or 30 mph. So Annie must travel at 20 mph and Bonnie must travel at 10 mph. After 30 minutes Annie has traveled 10 miles and Bonnie has traveled 5 miles so the distance between them is 15 miles. At 10 mph it will take Bonnie 1 1/2 hours or 90 minutes to travel 15 miles.

3. Find the product of all real solutions to the equation $x^4 + 2x^2 - 35 = 0$.

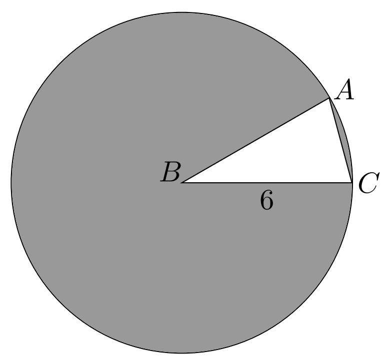
a) 5 b) -5 c) 7

d) -7 e) -35

Solution:

Correct answer: -5 (**b**)
$x^4 + 2x^2 - 35 = (x^2 - 5)(x^2 + 7) = 0$. If $x^2 - 5 = 0$, $x = \pm\sqrt{5}$. If $x^2 + 7 = 0$, there are no real solutions. The product of the real solutions is -5.

4. In the following diagram, which is not drawn to scale, the shaded area is 32π. The circle has radius $\overline{BC} = \overline{BA} = 6$. Find the measure of $\angle ABC$.



a) $30°$ b) $40°$ c) $\arcsin\left(\frac{2\pi}{9}\right)$

d) $\arcsin\left(\frac{\sqrt{2}}{3}\right)$ e) $45°$

Solution:

Correct answer: $\arcsin\left(\frac{2\pi}{9}\right)$ (**c**)
The area of the circle is 36π.
The area of the triangle is $4\pi = (1/2)6 \cdot 6 \sin(< ABC)$.
So $\sin(< ABC) = 2\pi/9$. Thus the angle is $\arcsin(2\pi/9)$.

5. Traveling from Salt Lake City to Denver, a regularly priced adult ticket is discounted by 15% for seniors and 50% for children. Tickets for a party of 3 seniors, 5 adults, and 7 children cost \$884. How much will it cost in dollars for 2 seniors, 6 adults, and 8 children?

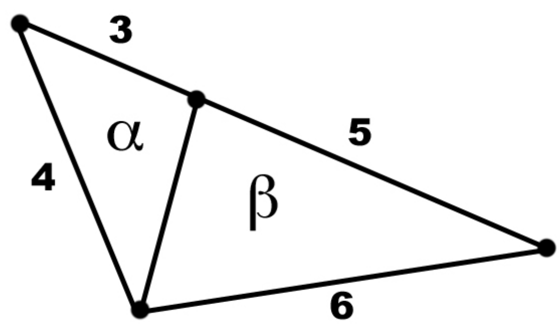
a) 884 b) 856 c) 868

d) 936 e) 976

Solution:

Correct answer: 936 (**d**)
Let x be the cost in dollars for an adult fare. Then the fare for a senior in dollars is $.85x = \frac{17x}{20}$ and the fare for a child in dollars is $.5x = \frac{x}{2}$. For a party of 3 seniors, 5 adults, and 7 children the cost in dollars is $3 \cdot \frac{17x}{20} + 5x + 7 \cdot \frac{x}{2} = 884$. Multiplying both sides by 20 and simplifying yields $51x + 100x + 70x = 221x = 20 \cdot 884 = 20 \cdot 4 \cdot 221$. So $x = 20 \cdot 4 = 80$. The fare for a senior is $0.85 \cdot \$80 = \$68$. The adult fare is \$80 and the child fare is $0.5 \cdot \$80 = \$40$. The cost in dollars for 2 seniors, 6 adults, and 8 children is $2 \cdot 68 + 6 \cdot 80 + 8 \cdot 40 = 136 + 480 + 320 = 936$.

6. The large triangle below is divided into two triangles of areas α and β. Find α/β.



a) $\frac{3}{5}$ b) $\frac{1}{2}$ c) $\frac{4}{6}$

d) $\frac{3}{4}$ e) $\frac{7}{11}$

Solution:

Correct answer: $\frac{3}{5}$ (**a**)
Over bases 3 and 5, the two triangles have the same altitude.

7. Use properties of logarithms to find the exact value of the expression

$$\log_5 2 \cdot \log_2 125.$$

a) 3 b) 2 c) 1

d) 0 e) -1

Solution:

Correct answer: 3 (**a**)

$$\log_5 2 \cdot \log_2 125 = \frac{\log_2 2}{\log_2 5} \log_2 5^3 = \frac{1}{\log_2 5} 3 \log_2 5 = 3$$

8. Let

$$f(x) = \begin{cases} 2x & \text{for} \quad 0 \le x \le 0.5 \\ 2(1-x) & \text{for} \quad 0.5 < x \le 1. \end{cases}$$

If $x_0 = \frac{6}{7}$ and $x_n = f(x_{n-1})$ for $n \ge 1$, find x_{100}.

a) $\frac{2}{7}$ b) $\frac{4}{7}$ c) $\frac{6}{7}$

d) $\frac{10}{7}$ e) $\frac{96}{7}$

Solution:

Correct answer: $\frac{2}{7}$ (**a**)
$x_0 = 6/7, x_1 = 2/7, x_2 = 4/7, x_3 = 6/7, x_4 = 2/7, x_5 = 4/7.$
The sequence repeats every three terms.
$x_{100} = x_{97} = \cdots = x_1 = 2/7.$

9. When $3x^{12} - x^3 + 5$ is divided by $x + 1$ the remainder is:

a) 1 b) 3 c) 5

d) 7 e) 9

Solution:

Correct answer: 9 (**e**)
By the Remainder Theorem, the answer is the polynomial evaluated at x = -1.
$3(-1)^{12} - (-1)^3 + 5 = 3 + 1 + 5 = 9.$

10. A factor of $243,000,000$ is chosen at random. What is the probability that the factor is a multiple of 9?

a) 0 b) $\frac{1}{6}$ c) $\frac{1}{3}$

d) $\frac{2}{3}$ e) $\frac{1}{3,000}$

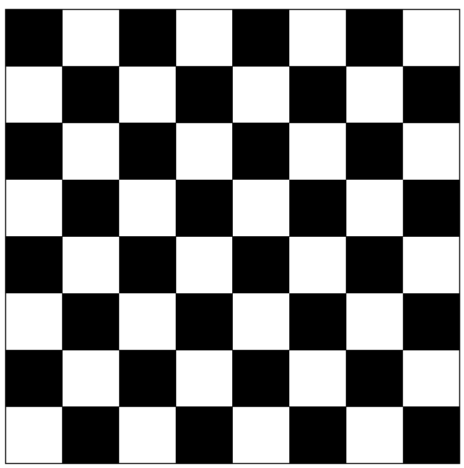
Solution:

Correct answer: $\frac{2}{3}$ (**d**)

$243,000,000 = 2^6 3^5 5^6$. A factor of $243,000,000$ is the form $3^n \cdot K$, where $n = 0, 1, 2, 3, 4,$ or 5 and K is not divisible by 3. The factor is a multiple of 9 exactly when $n = 2, 3, 4,$ or 5. The probability of this is $\frac{4}{6} = \frac{2}{3}$.

11. How many different kinds of pieces can you cut from an 8×8 checkered board consisting of four 1×1 squares that are joined end to end?

Note: Two pieces are the same if one of the pieces can be rotated or translated in the plane to obtain the other piece.

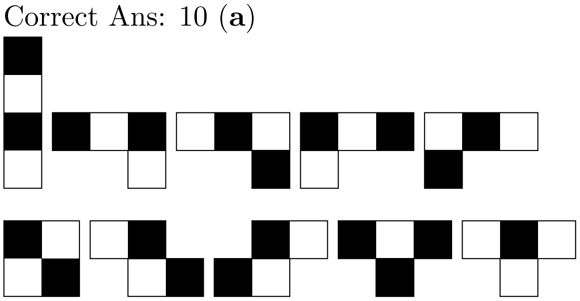


a) 10 b) 12 c) 20

d) 8 e) 16

Solution:

Correct Ans: 10 (**a**)



12. Sally decides to hike to the Y on Y-mountain. The first part of her hike (from her home to the trailhead) was on level ground. On level ground Sally walks at a rate of 3.5 miles per hour. From the trailhead to the Y she hikes at an average rate of 2.4 miles per hour. From the Y back to the trailhead she hikes at about 4.2 miles per hour. Finally, she returns home (once again at a rate of 3.5 miles per hour). Given that two hours have passed between when she leaves home and when she returns and that the total round trip distance is 6.65 miles, how long is the hike from the trailhead to the Y?

a) .6 miles
b) .7 miles
c) 1.2 miles
d) 1.4 miles
e) 2.1 miles

Solution:

Correct answer: 1.2 miles (**c**)
Let x be the distance in miles from the trailhead to the Y, then the distance from home to the trailhead in miles is $\frac{6.65}{2} - x = \frac{6.65 - 2x}{2}$. The time in hours on flat ground is $\frac{6.65 - 2x}{3.5}$. In hours, the time going up the mountain is $\frac{x}{2.4}$ and the time coming back is $\frac{x}{4.2}$. Since the total time is 2 hours, we need to solve the equation

$$\frac{6.65 - 2x}{3.5} + \frac{x}{2.4} + \frac{x}{4.2} = 2.$$

Converting the denominators to fractions yields

$$\frac{13.3 - 4x}{7} + \frac{5x}{12} + \frac{5x}{21} = 2.$$

Multiplying both sides by 84 gives

$$159.6 - 48x + 35x + 20x = 168 \Rightarrow 7x = 8.4 \Rightarrow x = 1.2.$$

13. How many ways can you write 5 as the sum of one or more positive integers if different orders are not counted differently? For example, there are three ways to write 3 in this way: 1 + 1 + 1, 1 + 2, and 3.

a) 7
b) 6
c) 8
d) 5
e) 10

Solution:

Correct answer: 7 (**a**)
5,
4 + 1,
3 + 2,
3 + 1 + 1,
2 + 2 + 1,
2 +1 + 1 + 1,
1 + 1 + 1 +1 + 1.

14. How many real solutions does the equation $x^{3/2} - 32x^{1/2} = 0$ have?

a) 0 b) 1 c) 2

d) 3 e) 4

Solution:

Correct answer: 2 (**c**)
Let $y = x^{1/2} = \sqrt{x}$. the original equation becomes $y^3 - 32y = 0$ or $y(y^2 - 32) = 0$. Thus $y = 0$ or $y = \pm\sqrt{32} = \pm 4\sqrt{2}$. If $y = \sqrt{x} = 0$, then $x = 0$. If $y = \sqrt{x} = \sqrt{32}$, then $x = 32$. There is no real value of x for which $y = \sqrt{x} = -\sqrt{32}$, since for x real $\sqrt{x} \geq 0$. So there are two real solutions to the equation.

15. When slicing a rectangular cake, what is the smallest number of straight cuts that you need to make exactly 7 pieces?

a) 6 b) 5 c) 7

d) 3 e) 4

Solution:

Correct answer: 3 (**d**)
One straight cut gives two pieces. A second straight cut gives 1 or 2 additional pieces depending on whether the second cut crosses the first cut. The third straight cut gives 1, 2, or 3 additional pieces depending on whether it crosses previous cuts and misses the intersection of previous cuts. So it is possible to get 7 pieces with three straight cuts as long as the second cut meets the first cut, and the third cut meets the previous two cuts in distinct points.

16. A very thin disk has an area (on one side) of 4π. A square window is cut into a wall. What is the smallest area (the *lower bound*) the window can have and still be large enough for the disk to fit through?

a) $16/\pi$ b) 4π c) $8\sqrt{2}$

d) 8 e) 16

Solution:

Correct answer: 8 (**d**)
If the disk is put in sideways through the diagonal. The disk has radius 2, so a diameter has length 4. A square with diagonal of length 4 has sides of length $2\sqrt{2}$, and area 8.

17. How many whole numbers from 1 to 10000, inclusive, are multiples of 20 but not of 22?

a) 489 b) 478 c) 455

d) 458 e) 432

Solution:

Correct answer: 455 (**c**)
There are 500 multiples of 20, But 45 of them are simultaneously multiples of 11.

18. Find the sum of all the fractions strictly between 0 and 1 which, in reduced form, have denominator less than or equal to 10.

a) $\frac{21}{6}$ b) $\frac{43}{4}$ c) 25

d) $\frac{43}{2}$ e) $\frac{31}{2}$.

Solution:

Correct answer: $\frac{31}{2}$ (**e**)
For denominators 2,3,4,5,6,7,8,9,10 respectively, the sums are $\frac{1}{2}$, 1,1,2,1,3,2,3,2 respectively.

19. Express as single complex number: $1 + i + i^2 + i^3 + ... + i^{100}$ where $i^2 = -1$

a) i b) -1 c) 1

d) $100i$ e) 0

Solution:

Correct answer: 1 (**c**)
First notice that
$1 + i + i^2 + i^3 = 1 + i + (-1) + (-i) = 0$
Similarly for any k:
$i^{4k} + i^{4k+1} + i^{4k+2} + i^{4k+3} = 0$
Thus,
$1 + i + i^2 + i^3 + ... + i^{96} + i^{97} + i^{98} + i^{99} + i^{100} = i^{100}$ and
$i^{100} = (\sqrt{-1})^{100} = (-1)^{50} = 1$.

20. The graph of the function $h(x)$ is a straight line. On the interval $2 \leq x \leq 4$ the function $h(x)$ satisfies $h(x) = 3 + |x-4| + 2|x-6|$. What is $h(7)$?

a) -5

b) -2

c) 0

d) 3

e) 19

Solution:

Correct answer: -2 (**b**)
$h(x) = -3x + 19$ on $[2, 4]$.

21. Let $\epsilon = 10^{-25} = .0000000000000000000000001$, and let $x = \sqrt{1+2\epsilon}$, $y = \sqrt[3]{1+3\epsilon}$ and $z = 1+\epsilon$. Rank the numbers x, y and z. If any of them are equal, say so.

a) $y > x > z$

b) $x < y < z$

c) $x = y = z$

d) $y < x < z$

e) $x = y > z$

Solution:

Correct answer: $y < x < z$ (**d**)
By squaring z, one can see it is greater than x. By taking x and y to the sixth power (for example) one can see that $x > y$.

22. How many of the following triples can be the side lengths of an obtuse triangle?

$$(2,2,3), (3,5,7), (3,7,11), (7,9,11)$$

a) 0

b) 1

c) 2

d) 3

e) 4

Solution:

Correct answer: 2 (**c**)
For a triangle to exist, the length of the longest side must be less than the sum of the lengths of the other two sides. If (a, b, c) gives the side lengths of a triangle, the Pythagorean Theorem says that the angle between the sides of length a and b is a right angle if and only if $c^2 = a^2 + b^2$. If $c^2 < a^2 + b^2$, then the angle between the sides of length a and b is an acute angle. If $c^2 > a^2 + b^2$, then the angle between the sides of length a and b is an obtuse angle. The first two triples give the side lengths of obtuse triangles. The third triple does not give the side lengths of any triangle. The last triple gives the side lengths for an acute triangle.

23. Seven students in a classroom are to be divided into two groups of two and one group of three. In how many ways can this be done?

a) 3   b) 35   c) 90

d) 105   e) 315

Solution:

Correct answer: 105 (**d**)

If the group of three is chosen first, this can be done $\binom{7}{3} = 35$ ways.

The remaining 4 students can be divided into pairs 3 different ways. So there are $35 \cdot 3 = 105$ ways.

24. If $|r| < 1$, then $(a)^2 + (ar)^2 + (ar^2)^2 + (ar^3)^2 + \cdots =$

a) $\dfrac{a^2}{(1-r)^2}$ b) $\dfrac{a^2}{1+r^2}$ c) $\dfrac{a^2}{1-r^2}$

d) $\dfrac{4a^2}{1+r^2}$ e) none of these

Solution:

Correct answer: $\dfrac{a^2}{1-r^2}$ (**c**)

This is a geometric series with first term a^2 and ratio r^2. The sum of an infinite geometric series with ratio of absolute value less than 1 is the first term divided by one minus the ratio; i.e., $\dfrac{a^2}{1-r^2}$.

25. When buying a bike from the *Math Bikes* company, there are three extra options to choose (a bell, a rear fender, and a basket), each of which you can choose to add to the bike or choose not to add it. If *Math Bikes* has sold 300 bikes, what is the largest number of bikes that you can guarantee to have exactly the same extras as each other?

a) 8   b) 37   c) 38

d) 43   e) 292

Solution:

Correct answer: 38 (**c**)

There are $2^3 = 8$ possible bike types. Now $300 \div 8 = 37.5$. If there were 37 or less of each type of bike, then there would be less than 300 bikes. So there must be at least 38 bikes that have exactly the same extras.

26. Begin with 63, and keep repeating the following pair of operations: *Add 1, then take the square root.* Thus we generate the following sequence of numbers: $63, 8, 3, 2, \sqrt{3}, \sqrt{1+\sqrt{3}}$, etc. Eventually, those numbers settle down to a *limit*. What is the limit?

a) $1 + \dfrac{\sqrt{2}}{3}$

b) $\dfrac{1+\sqrt{5}}{2}$

c) $\sqrt{1+\sqrt{2}}$

d) $\dfrac{1+\sqrt{2}}{2}$

e) 1

Solution:

Correct answer: $\dfrac{1+\sqrt{5}}{2}$ (**b**)
As the numbers settle down to the limit, each number is very close to its successor. In the limit, we may assume each number equal to its successor, thus $x = \sqrt{x+1}$. That leads to the equation $x^2 = x + 1$ which can be solved by quadratic formula to give answer (b).

27. A square and an equilateral triangle have the same area. Let A be the area of the circle circumscribed around the square and B be the area of the circle circumscribed around the triangle. Find $\dfrac{A}{B}$.

a) $\dfrac{3\sqrt{3}}{8}$

b) $\dfrac{3\sqrt{3}}{6}$

c) $\dfrac{3\sqrt{3}}{4}$

d) $\dfrac{3}{8}$

e) $\dfrac{3}{4}$

Solution:

Correct answer: $\dfrac{3\sqrt{3}}{8}$ (**a**)
The $\mathcal{A}$ be the common area. Let e be the edge length of the square, then $\mathcal{A} = e^2$. Let s be the side length of the triangle, then $\mathcal{A} = \dfrac{s^2\sqrt{3}}{4}$.
The center of the square is the point where the two diagonals meet. The radius r of the circle circumscribed around the square is the distance from the center of the square to the four vertices which is $\dfrac{e}{\sqrt{2}}$.
The circumcenter of the triangle is the point where the perpendicular bisectors meet which, in this case, is the same as the centroid or the point where the medians meet. The centroid lies on the medians two-thirds the distance from the vertex to the midpoint of the opposite side. The length of a median is $\dfrac{s\sqrt{3}}{2}$. So the radius R of the circle circumscribed about the triangle is $\dfrac{2}{3} \cdot \dfrac{s\sqrt{3}}{2} = \dfrac{s}{\sqrt{3}}$

$$\frac{A}{B} = \frac{\pi r^2}{\pi R^2} = \frac{r^2}{R^2} = \frac{e^2}{2} \div \frac{s^2}{3} = \frac{\mathcal{A}}{2} \div \frac{4\mathcal{A}}{3\sqrt{3}} = \frac{3\sqrt{3}}{8}.$$

28. Find the number of diagonals that can be drawn in a convex polygon with 200 sides.

Note: A *diagonal* of a polygon is any line segment between non-adjacent vertices.

a) 1,969 b) 1,970 c) 20,000

d) 19,700 e) 19,699

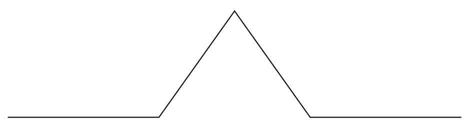
Solution:

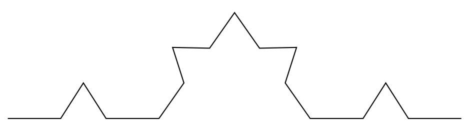
Correct answer: 19,700 (**d**)
Each vertex has 197 diagonals, but each diagonal gets counted twice. $200 \cdot 197/2 = 100 \cdot 197 = 19,700$.

29. Koch's curve is created by starting with a line segment of length one. Call this stage 1.

Stage 1

To get from one stage to the next we divide each line segment into thirds and replace the middle third by two line segments of the same length.

Stage 2 

Stage 3 

What is the length of Koch's curve at Stage 6?

a) 16/9 b) 16/27 c) 16/81

d) 1024/81 e) 1024/243

Solution:

Correct answer: 1024/243 (**e**)
Stage 1 has length 1. Stage $n+1$ is 4/3 as long as Stage n. Stage 6 the answer would be $(4/3)^5 = 1024/243$.

30. For a certain baseball team the probability of winning any game is P, (the probability of winning a particular game is independent of any other games). What is the probability the team wins 3 out of 5 games?

a) $10P^2(1-P)^3$ b) $10P^3(1-P)^2$ c) $5P^3(1-P)^2$

d) $5P^2(1-P)^3$ e) $P^3(1-P)^2$

Solution:

Correct answer: $10P^3(1-P)^2$ (**b**)
The number of possible outcomes from playing 5 games is 2^5. There are 5 choose 3 or 10 ways of winning 3 games and losing 2; e.g. winning the first three and losing the last two, winning the first two and fourth game and losing the other two, etc. Each of these ways of winning 3 and losing 2 games has probability $P^3(1-P)^2$. So the total probability of winning 3 games and losing 2 is $10P^3(1-P)^2$.

31. If x is the *fraction* of numbers between 1 and 1,000, inclusive, which contain 4 as a digit, and y is the *fraction* of numbers between 1 and 10,000, inclusive which contain 4 as a digit, what is x/y?

a) 2/3 b) 3/4 c) 27/34

d) 271/3439 e) 2710/3439

Solution:

Correct answer: 2710/3439 (**e**)
We can count how many numbers in $\{1, 2, \dots 1,000\}$ contain 4 as a digit using the inclusion exclusion principle: If one digit is a 4, there are 10^2 possiblities for the other three digits, and we can do this allowing the 4 to be in the ones, the tens, and the hundreds digit, but we've overcounted. We need to subtract off the number of ways to get *two* 4's. But then we've undercounted, and we need to add back the number of ways to get *three* 4's:

$$3 \cdot (100) - 3 \cdot (10) + 1 = 271.$$

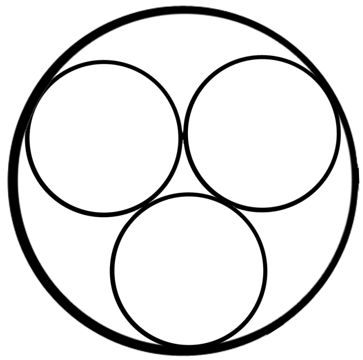
So, $x = 271/1000$. We calculate y similarly: there are

$$4 \cdot (1,000) - 6(100) + 4(10) - 1 = 3439$$

numbers in $\{1, 2, \dots, 10,000\}$ containing 4 as a digit, so $y = 3439/10,000$. Thus,

$$\frac{x}{y} = \frac{\frac{271}{1,000}}{\frac{3439}{10,000}} = \frac{2710}{3439}.$$

32. Given that the area of the outer circle is ten square units, find the area of any one of the three equal circles which are tangent to each other and to the outer circle, and inscribed inside the circle of ten square units.



a) $30(7-4\sqrt{3})$ square units.

b) 2.5 square units.

c) $\dfrac{10}{(3+\sqrt{2})}$ square units.

d) $\sqrt[3]{10}$ square units.

e) 2 square units.

Solution:

Correct answer: $30(7-4\sqrt{3})$ square units (**a**)
The radius of the outer circle is $R = r + r\csc(\pi/3)$ where r is the radius of an inner circle.

33. Find the product of the zeros of $z^8 + 4z^4 + 16$ that lie in the first quadrant of the complex plane.

a) $\sqrt{3}$

b) $\sqrt{3}i$

c) $1+i$

d) 2

e) $2i$

Solution:

Correct answer: $2i$ (**e**)
Notice that $(z^4-4)(z^8+4z^4+16) = z^{12}-64 = z^{12}-2^6$. The zeros of z^4-4 are $\pm\sqrt{2}$ and $\pm\sqrt{2}i$. By De Moivres Theorem, the zeros of $z^{12}-64$ are $z = \sqrt{2}(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6})$ for $n = 0, 1, 2, \ldots, 11$. So the zeros of z^8+4z^4+16 that lie in the first quadrant are $\sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{2}(\frac{\sqrt{3}}{2} + \frac{1}{2}i)$ and $\sqrt{2}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = \sqrt{2}(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$.

34. The natives of Wee-jee Islands rate 2 spears as worth 3 fishhooks and a knife, and will give 25 coconuts for 3 spears, 2 knives, and a fishhook together. Assuming each item is worth a whole number of coconuts, how many coconuts will the natives give for each article separately?

a)

Item	Worth in Coconuts
fishhook	1
knife	3
spear	3

b)

Item	Worth in Coconuts
fishhook	1
knife	5
spear	4

c)

Item	Worth in Coconuts
fishhook	2
knife	2
spear	4

d)

Item	Worth in Coconuts
fishhook	2
knife	4
spear	5

e)

Item	Worth in Coconuts
fishhook	3
knife	3
spear	6

Solution:

Correct Ans: (**d**)

Let $S, F,$ and K be the value of a spear, knife and fishhook in coconuts. Then

$$2S = 3F + K$$

and

$$25 = 3S + 2K + F.$$

Multiply the second equation above by 2 and the first equation by 3 to find

$$50 = 6S + 4K + 2F$$

$$6S = 9F + 3K.$$

Substituting the second equation into the first gives:

$$50 = 11F + 7K.$$

We know that F and K are nonnegative integers. If $F \geq 5$ then $11F + 7K \geq 55$, hence $F = 0, 1, 2, 3,$ or 4. We try each of these four cases, and only $F = 2$, $K = 4$ works, and in this case $S = 5$.

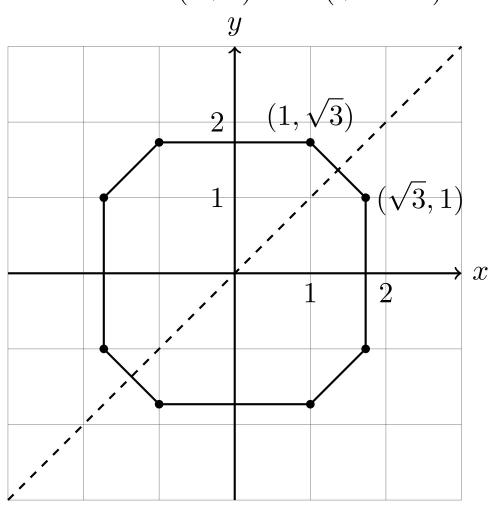
35. An octagon in the plane is symmetric about the x-axis, the y-axis, and the line whose equations is $y = x$. If $(1, \sqrt{3})$ is a vertex of the octagon, find its area.

a) $6\sqrt{3}$ b) 11 c) $6 + 2\sqrt{3}$

d) $2 + 6\sqrt{3}$ e) $4 + 4\sqrt{3}$

Solution:

Correct answer: $4 + 4\sqrt{3}$ (**e**)
The vertices of the octagon are $(\pm 1, \pm\sqrt{3})$ and $(\pm\sqrt{3}, \pm 1)$. The area can be computed by computing the area of the enclosing square that is $2\sqrt{3} \times 2\sqrt{3}$ and subtracting the area of the four right isosceles triangles or, equivalently, the area of two squares that are $\sqrt{3} - 1 \times \sqrt{3} - 1$. The area is $(2\sqrt{3})^2 - 2(\sqrt{3} - 1)^2 = 12 - 2(4 - 2\sqrt{3}) = 4 + 4\sqrt{3}$.



36. A regular octahedron is formed by setting its vertices at the centers of the faces of the cube. Another regular octahedron is formed around the cube by making the center of each triangle of the octahedron hit at a vertex of the cube. What is the ratio of the volume of the larger octahedron to that of the smaller octahedron?

a) $2\sqrt{2}$ b) $27/8$ c) $3\sqrt{3}$

d) 8 e) 27

Solution:

Correct answer: 27 (**e**)
If the cube is centered at the origin with x,y,z three dimensional coordinates, then the vertices of the cube can be at $(\pm a, \pm a, \pm a)$. The vertices of the smaller octahedron are $(\pm a, 0, 0), (0, \pm a, 0), (0, 0, \pm a)$. The vertices of the larger octahedron are $(\pm 3a, 0, 0), (0, \pm 3a, 0), (0, 0, \pm 3a)$ since the centroid of the triangle with vertices $(3a, 0, 0), (0, 3a, 0), (0, 0, 3a)$ is (a, a, a). Since the larger octahedron is similar to the smaller by a scale factor of 3, the volume is $3^3 = 27$ times as large.

37. A square number is an integer number which is the square of another integer.
Positive square numbers satisfy the following properties:

- The units digit of a square number can only be 0, 1, 4, 5, 6, or 9.
- The digital root of a square number can only be 1, 4, 7, or 9.
  The *digital root* is found by adding the digits of the number. If you get more then one digit you add the digits of the new number. Continue this until you get to a single digit. This digit is the digital root.

One of the following numbers is a square. Which one is it?

a) $4,751,006,864,295,101$

b) $3,669,517,136,205,224$

c) $2,512,339,789,576,516$

d) $1,898,732,825,398,318$

e) $5,901,643,220,186,107$

Solution:

Correct Ans: $2,512,339,789,576,516$ (**c**)
(d), and (e) are ruled out because of the first property.
(a) and (b) are rulld out because their digit roots are 5 and 8.
(c) has a digit root of 7 thus it is the square number.

38. Let $f(x) = 9x^2 + dx + 4$. For certain values of d, the equation $f(x) = 0$ has only one solution. For such a value of d, which value of x could be a solution to $f(x) = 0$?

a) $\frac{2}{3}$ b) 1 c) $\frac{4}{3}$

d) 3 e) 12

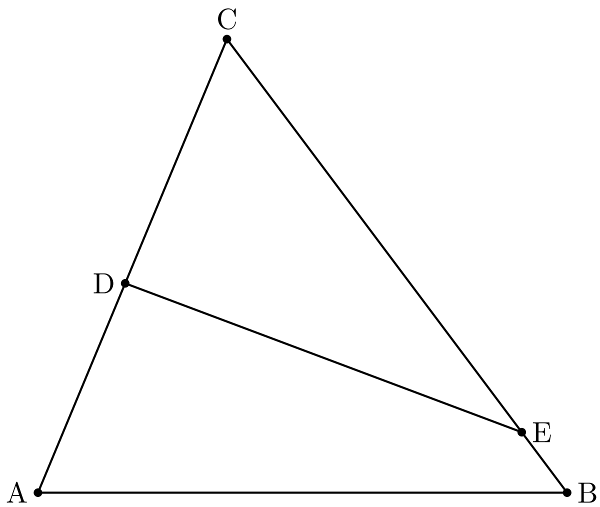
Solution:

Correct answer: $\frac{2}{3}$ (**a**)
Using the quadratic equation gives $d = \pm 12$ as the only values for one real solution.

39. In $\triangle ABC$, $AC = 13$, $BC = 15$ and the area of $\triangle ABC = 84$. If $CD = 7$, $CE = 13$, and the area of $\triangle CDE$ can be represented as $\frac{p}{q}$ where p and q are relatively prime positive integers, find q.

a) 3 b) 5 c) 7

d) 11 e) 13



Solution:

Correct answer: 5 (**b**)

The area of $\triangle ABC$ is $\frac{1}{2} \cdot AC \cdot BC \cdot \sin C = \frac{1}{2} \cdot 13 \cdot 15 \cdot \sin C = 84$. The area of $\triangle CDE$ is $\frac{1}{2} \cdot CD \cdot CE \cdot \sin C = \frac{1}{2} \cdot 7 \cdot 13 \cdot \sin C$. Let $[ABC]$ be the area of $\triangle ABC$ and $[CDE]$ be the area of $\triangle CDE$. Then $\frac{[CDE]}{[ABC]} = \frac{7 \cdot 13}{13 \cdot 15} = \frac{7}{15}$. So $[CDE] = \frac{7}{15}[ABC] = \frac{7}{15}84 = \frac{7 \cdot 28}{5} = \frac{196}{5}$.

40. Consider the sequence $1, 9, 5, 7, 6, \frac{13}{2}, \frac{25}{4}, \ldots$, where each element in the sequence is the *average* of the preceeding two. What is the largest real number smaller than infinitely many elements of the sequence?

a) 7 b) $\frac{19}{3}$ c) $\frac{31}{5}$

d) $\frac{20}{3}$ e) 6

Solution:

Correct answer: $\frac{19}{3}$ (**b**)

One can show that each number is half as far from $\frac{19}{3}$ as the previous, but on the opposite side. Thus half of the numbers are below the answer, half above, and they get arbitrarily close.

END OF EXAM