2022 State Math Competition

Senior Exam

Version A

Instructions:

- Make sure to mark the version on your answer sheet.
- Correct answers are worth 5 points. Unanswered questions will be given 2 points. Incorrect answers will be worth 0 point. This means that it is not in your best interest to guess answers unless you have eliminated some possibilities.
- Fill in the answers on the answer sheet using a pencil or pen.
- Time limit: 75 minutes.
- When you are finished, please give the exam and any scrap paper to the test administrator.
- Good luck!

1. Solve for x:

$$\log_5\left(\log_2\left(\log_3\left(2x-3\right)\right)\right)=0$$

(A) 0
(B) 6
(C) 1
(D) 3
(E) 9

Correct answers: (B)

Explanation: $\log_5(y)=0 \Rightarrow y=1$, which implies that $1=\log_2\left(\log_3\left(2x-3\right)\right)$. Now if $1=\log_2(y)\Rightarrow y=2$ which implies that $2=\log_3\left(2x-3\right)$. Now if $\log_3(y)=2$, then $y=3^2=9$. So we get our solution by solving $2x-3=9$, or $x=6$.

2. For a positive integer n, let $f(n)$ be the number of positive factors of n. For example, since 6 has factors 1, 2, 3, and 6, $f(6)=4$. What is $f(f(f(24)))$?

(A) 4
(B) 2
(C) 8
(D) 3
(E) 1

Correct answers: (D)

Explanation: Since 24 has factors 1, 2, 3, 4, 6, 8, 12, and 24, $f(24)=8$. The factors of 8 are 1, 2, 4, and 8, so $f(f(24))=f(8)=4$. The factors of 4 are 1, 2, and 4, so $f(f(f(24)))=f(f(8))=f(4)=3$.

3. Which of the following shows the numbers $2^{1/2}, 3^{1/3}, 6^{1/6}$ in increasing order?

(A) $2^{1/2}<3^{1/3}<6^{1/6}$
(B) $3^{1/3}<6^{1/6}<2^{1/2}$
(C) $6^{1/6}<3^{1/3}<2^{1/2}$
(D) $3^{1/3}<2^{1/2}<6^{1/6}$
(E) $6^{1/6}<2^{1/2}<3^{1/3}$

Correct answers: (E)

Explanation: We raise these three numbers to the 6th power.

- $(2^{1/2})^6 = 2^3 = 8$
- $(3^{1/3})^6 = 3^2 = 9$
- $(6^{1/6})^6 = 6$

And note given $a, b > 0$, then $a < b$ if and only if $a^6 < b^6$, so the answer for this problem is $6^{1/6} < 2^{1/2} < 3^{1/3}$.

4. Given that $i^2 = -1$, simplify $(\sqrt{3} + i)^9$.

(A) 0
(B) 512
(C) -512
(D) $512i$
(E) $-512i$

Correct answers: (E)
Explanation: $\sqrt{3} + i = 2e^{i\pi/6}$. So $(\sqrt{3}+i)^9 = 2^9 e^{i3\pi/2} = -2^9 i = -512i$

5. Define a piecewise function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{2}x & x \leq 2 \\ 2x - 3 & x \geq 2 \end{cases}.$$

The function f is bijective (one-to-one and onto). What is the inverse function to f?

(A) $f^{-1}(y) = \begin{cases} 2y & y \leq 4 \\ \frac{1}{2}y - \frac{3}{2} & y \geq 4 \end{cases}$
(B) $f^{-1}(y) = \begin{cases} 2y & y \leq 2 \\ \frac{1}{2}y + \frac{3}{2} & y \geq 2 \end{cases}$
(C) $f^{-1}(y) = \begin{cases} 2y & y \leq 1 \\ \frac{1}{2}y + 3 & y \geq 1 \end{cases}$
(D) $f^{-1}(y) = \begin{cases} 2y & y \leq 1 \\ \frac{1}{2}y + \frac{3}{2} & y \geq 1 \end{cases}$
(E) None of the above

Correct answers: (D)
Explanation: Taking $y = f(x)$ we have $y \leq 1$ if and only if $x \leq 2$ and there solving $y = 2x$ gives $x = f^{-1}(y) = \frac{1}{2}y$. We have $y \geq 1$ if and only if $x \geq 2$ and there solving $y = 2x - 3$ gives $x = f^{-1}(y) = \frac{1}{2}(y + 3)$.

6. At 12:00 PM two ships set out from shore. Ship A travels due east at 3 miles per hour. Ship B begins exactly one mile to the north of Ship A and travels exactly $45°$ due north-east at $\sqrt{2}$ miles per hour. How quickly in miles per hour is the distance between Ship A and Ship B changing at 1:00 PM?

(A) $\frac{11}{\sqrt{13}}$
(B) $\frac{3}{\sqrt{2}}$
(C) $\frac{5}{\sqrt{2}}$
(D) $\frac{3}{\sqrt{5}}$
(E) None of the above

Correct answers: (B)

Explanation: Let's parametrize the ships using points in the plane. Say that A starts out at $(0,0)$. Then its position at time t is $(3t,0)$. The position of B is $(t,1+t)$. Thus the square of the distance between them is $D^2=(3t-t)^2+(1+t)^2$. So we have

$$D=\sqrt{4t^2+(1+t)^2}=\sqrt{5t^2+2t+1}.$$

We have

$$\frac{dD}{dt}=\frac{1}{2}\frac{10t+2}{\sqrt{5t^2+2t+1}}=\frac{5t+1}{\sqrt{5t^2+2t+1}}.$$

For $t=1$ we have $\frac{dD}{dt}=\frac{6}{\sqrt{8}}=\frac{3}{\sqrt{2}}$.

7. Which of the following numbers is the largest?

(A) the number of ways to arrange 3 red balls and 2 blue balls in a line
(B) the number of ways to choose 3 items out of a set of 5 distinct items
(C) the number of ways to put 3 distinct items into 5 boxes
(D) the number of ways to order five distinct items
(E) the number of ways to put 5 distinct items into 3 boxes

Correct answers: (E)

Explanation: The quantities are:

- the number of ways to choose 3 items out of a set of 5 distinct items is $\binom{5}{3}=10$
- the number of ways to put 3 distinct items into 5 boxes is $5^3=125$
- the number of ways to put 5 distinct items into 3 boxes $3^5=243$
- the number of ways to order five distinct items is $5!=120$
- the number of ways to arrange 3 red balls and 2 blue balls in a line is $\binom{5}{3}=10$. This can be seen because there are 5 possible positions to place each ball. An arrangement is determined by the 3 positions that the red balls are placed into.

8. How many integers n exist such that n^2+n+1 is a perfect square?

(A) 1
(B) infinitely many
(C) 3
(D) 2
(E) 0

Correct answers: (D)

Explanation: Indeed, if $n=0$ or -1, then $n^2+n+1=1$ is a perfect square. If $n>0$ then

$$n^2<n^2+n+1<(n+1)^2.$$

Thus n^2+n+1 is strictly between consecutive squares, so it cannot be a square itself. Similarly, if $n<-1$, then

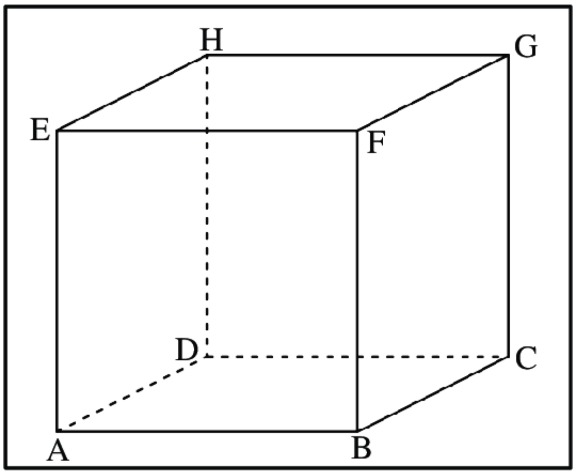
$$n^2+2n+1<n^2+n+1<n^2,$$

so there are no other numbers with this property.

9. A sphere is inscribed in a cube of edge length 1. Another (smaller) sphere is inscribed in the space between the larger sphere and one of the corners of the cube, so that it touches three faces of the cube, as well as the larger sphere. What is the radius of the smaller sphere?

(A) $1-\frac{\sqrt{3}}{2}$
(B) $\frac{1}{\sqrt{3}+1}$
(C) $\frac{1}{3}$
(D) $\frac{\sqrt{3}}{4}$
(E) $\sqrt{3}-1$

Correct answers: (A)



Explanation:

Denote the vertices of the cube by $ABCDEFGH$ (pictured). Let G be the vertex where we inscribe the smaller sphere. Denote by X the center of the smaller sphere. Let R and r denote the radius of the larger and the smaller sphere, respectively. Clearly, X lies on the diagonal AG. Furthermore, since the two spheres are touching, we can express the length $|AG|$ of the diagonal as

$$\frac{|AG|}{2} = R + r + |XG|. \tag{1}$$

One verifies easily (using the Pythagorean theorem) that $|AG| = \sqrt{3}$. Clearly $R = \frac{1}{2}$.

Now let Y be the orthogonal projection of X onto the face $BCGF$. Note that Y lies on BG. It follows that the triangle XYG is similar to ABG. In particular,

$$\frac{|XG|}{|XY|} = \frac{|AG|}{|AB|}.$$

We already know $|AB| = 1$, $|AG| = \sqrt{3}$. It remains to observe that $XY = r$, so that the above equation gives

$$|XG| = \sqrt{3}r.$$

Going back to (1), we get

$$\frac{\sqrt{3}}{2} = \frac{1}{2} + r(1+\sqrt{3}).$$

Solving for r, we obtain

$$r = \frac{\sqrt{3}-1}{2(\sqrt{3}+1)} = 1 - \frac{\sqrt{3}}{2}.$$

10. For real numbers x and y, define the operation $x \star y = x^2y^2$. Which of the following is false?

    (A) $x \star (y \star z) = (x \star y) \star z$, for all real x, y, z.
    (B) $(x \star y)^n = x^n \star y^n$, for all real x, y and integers n.
    (C) $x \star 0 = 0$, for all real x.
    (D) $x \star y = y \star x$ for all real x, y.
    (E) $x \star \frac{1}{x} = 1$, for all real $x \neq 0$.

Correct answers: (A)

Explanation: Here are the true statements:

$$x \star y = x^2y^2 = y^2x^2 = y \star x$$
$$(x \star y)^n = (x^2y^2)^n = x^{2n}y^{2n} = (x^n)^2(y^n)^2 = x^n \star y^n$$
$$x \star 0 = x^2 \cdot 0^2 = 0$$
$$x \star \frac{1}{x} = x^2 \cdot \frac{1}{x^2} = 1, x \neq 0$$

However

$$x \star (y \star z) = x \star (y^2z^2) = x^2(y^2z^2)^2 = x^2y^4z^4$$
$$(x \star y) \star z = (x^2y^2) \star z = (x^2y^2)^2z^2 = x^4y^4z^2$$

11. How many positive numbers x satisfy the equation $\cos(97x) = x$?

(A) 1
(B) 15
(C) 49
(D) 31
(E) 96

Correct answers: (D)

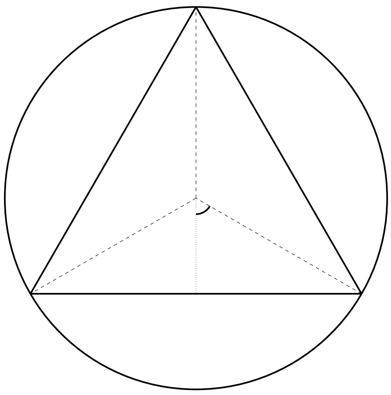
Explanation: Here because the cosine function only takes values between $[-1, 1]$, we only have to look at $x \in [0, 1]$. We note that $\frac{97/2\pi}{\approx}15$, which is approximated number of periods as x goes from 0 to 1. Note each period would give two solutions (if we ignore the last one or two), so the number of solution should be around 30. Thus the only possible choice is 31.

12. An equilateral triangle is inscribed in a circle of radius 1. What is the area of the triangle?

(A) $\sqrt{2}$
(B) $\frac{\sqrt{3}}{4}$
(C) $\frac{1}{2}$
(D) $\frac{\sqrt{2}}{2}$
(E) $\frac{3\sqrt{3}}{4}$

Correct answers: (E)

Explanation: Refer to the figure below. Placing line segments from the center of the triangle to the corners cuts it into three triangles. These line segments are radii of the circle so their lengths are all equal to 1. Thus all the dashed lines in the figure below have length 1. The angles between two consecutive dashed lines are $2\pi/3$. Consider one of the triangles cut out by the dashed lines and cut it into two right triangles. By trigonometry, the base of such a triangle is $\sin(\pi/3) = \sqrt{3}/2$ and the height is $1/2$. Thus, our inscribed equilateral triangle has base $2\frac{\sqrt{3}}{2} = \sqrt{3}$ and height $1 + \frac{1}{2} = \frac{3}{2}$. The area is $\frac{1}{2} \cdot \sqrt{3} \cdot \frac{3}{2}$.



13. Three different numbers are randomly selected out of the set of numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. What is the probability that their sum is greater than the sum of the remaining 6 numbers?

(A) $\frac{1}{42}$
(B) $\frac{1}{84}$
(C) $\frac{1}{14}$
(D) $\frac{3}{84}$
(E) None of the above

Correct answers: (A)

Explanation: The total number of possible choices of three numbers is $\binom{9}{3} = 84$. The sum of the numbers 1 through 9 is $\frac{1}{2}9 \cdot 10 = 45$. Thus, if the sum of three of the numbers is greater than the sum of the other six, then the three numbers must sum to a number greater than 22. The possibilities are 7, 8, and 9 (with a sum of 24) or 6, 8, and 9 (with a sum of 23). Thus the probability is $2/84 = 1/42$.

14. A number n may be written in base -2 as $a_k \cdots a_2 a_1 a_0$ where each digit a_i is 0 or 1. This means is that $n = a_k(-2)^k + \ldots + a_2(-2)^2 + a_1(-2)^1 + a_0$. For instance, 7 is 11011 when written in base -2. What is 200 written in base -2?

(A) 111101110
(B) 111011000
(C) 101101110
(D) 100110101
(E) None of the above

Correct answers: (B)

Explanation: We divide 200 by -2 repeatedly always keeping track of the remainder, which we insist to be 0 or 1.

$$\begin{array}{llll} 200 = (-100)(-2) & -100 = (50)(-2) & 50 = (-25)(-2) & -25 = (13)(-2) + 1 \\ 13 = (-6)(-2) + 1 & -6 = (3)(-2) & 3 = (-1)(-2) + 1 & -1 = (1)(-2) + 1 \end{array}.$$

We reassemble this information to compute the base -2 expansion as follows

$$\begin{array}{rclcl} 200 & = & (-100)(-2) & = & (50)(-2)^2 \\ & = & (-25)(-2)^3 & = & (13)(-2)^4 + (-2)^3 \\ & = & (-6)(-2)^5 + (-2)^4 + (-2)^3 & = & (3)(-2)^6 + (-2)^4 + (-2)^3 \\ & = & (-1)(-2)^7 + (-2)^6 + (-2)^4 + (-2)^3 & = & (-2)^8 + (-2)^7 + (-2)^6 + (-2)^4 + (-2)^3 \end{array}.$$

Thus the base -2 expansion is 111011000.

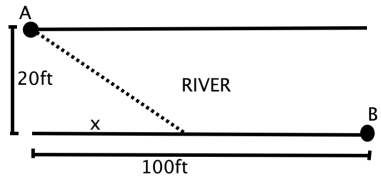
15. What is the remainder when 7^{111} is divided by 15?

(A) 7
(B) 13
(C) 1
(D) 4
(E) 9

Correct answers: (B)

Explanation: When $7^2 = 49$ is divided by 15 the remainder is 4. When 7^3 is divided by 15 the remainder is the remainder of $7 \cdot 4 = 28$ which is 13. When 7^4 is divided by 15 the remainder is the remainder of $7 \cdot 13 = 91$ which is 1. After this the remainders cycle through 7, 4, and 13 again. So a power 7^{4k} has remainder 1, a power 7^{4k+1} has remainder 7, a power 7^{4k+2} has remainder 4, and a power 7^{4k+3} has remainder 13. When 111 is divided by 4 the remainder is 3 (namely, $111 = 4 \cdot 27 + 3$) so 7^{111} has remainder 13.

16. You are on one bank of a slow-moving 20-foot wide river (labeled point A in the picture below). Suppose you want to get to point B on the other side of the river, 100 feet downstream. You want to do this as quickly as possible. You decide to first swim straight to a point x feet downstream on the opposite shore, then run the rest of the way to point B. If you run three times faster than you swim, what distance x downstream should you swim in order to get to point B as quickly as possible? Assume the current in the river is negligible.



(A) $\sqrt{\frac{80}{3}}$ feet
(B) $\sqrt{\frac{400}{3}}$ feet
(C) $\sqrt{80}$ feet
(D) $\sqrt{50}$ feet
(E) $\sqrt{\frac{400}{7}}$ feet

Correct answers: (D)

Explanation: Suppose you swim at s feet per second and run at $3s$ feet per second. Then if you land x feet downstream on the opposite shore, you travel time will be (using the fact that time is distance over speed):

$$T(x) = \frac{100 - x}{3s} + \frac{\sqrt{x^2 + 400}}{s}$$

To minimize this quantity, we differentiate and find critical points:

$$0 = T'(x) = -\frac{1}{3s} + \frac{x}{s\sqrt{x^2 + 400}}$$

or

$$\frac{1}{3} = \frac{x}{\sqrt{x^2 + 400}}$$

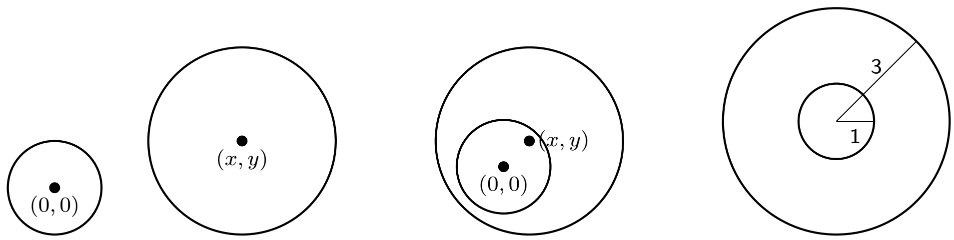
This has solution $x = \sqrt{50}$.

17. A frog is located at $(0, 0)$ in the plane. It makes a jump of length 1, meaning it jumps to a point distance 1 away from $(0, 0)$. Then it makes a second jump of length 2. What is the area of the region in the plane where the frog could land after the second jump?

(A) 8π
(B) 9π
(C) 3π
(D) 4π
(E) π

Correct answers: (A)

Explanation: After one jump, the frog lands on a point on the circle of radius 1 around $(0, 0)$. Consider a point (x, y) that can be reached after the two jumps. If we draw a circle of radius 2 around (x, y), then this circle must intersect the circle of radius 1 around $(0, 0)$. An intersection point is where the frog can land after the first jump, in order to jump a distance of 2 to (x, y). If (x, y) is more than 3 units from $(0, 0)$, then the circles cannot intersect because the sum of the radii is only 3 (left picture). Also if (x, y) is less than 1 unit from $(0, 0)$, the smaller circle is entirely inside the larger circle (middle picture). But when the distance is between 1 and 3 units, the circles will intersect. We want the area of the annulus centered at $(0,0)$ with inner radius 1 and outer radius 3 (right picture). We get $3^2 \cdot \pi - 1^2 \cdot \pi = 8\pi$.



18. A set of consecutive positive integers starting with 1 is written on a board. Then one number is erased. The average (arithmetic mean) of the remaining numbers is $\frac{135}{11} = 12 + \frac{3}{11}$. What number was erased?

    (A) cannot be determined
    (B) 8
    (C) 4
    (D) 6
    (E) 2

    Correct answers: (D)

    Explanation: Say initially the numbers $1, 2, \ldots, n$ were on the board. Their sum is $\frac{n(n+1)}{2}$. Let j be the number that was erased. Then the average of the remaining numbers is

$$\frac{\frac{n(n+1)}{2} - j}{n-1} = \frac{135}{11} = 12 + \frac{3}{11}.$$

Since the numerator of the left-hand side is an integer, we see that $n-1$ must be a multiple of 11. If $n - 1 = 11$, then $n = 12$. But then the average $12 + \frac{3}{11}$ is higher than the largest number on the board, which is impossible. If $n - 1 = 22$, then $n = 23$. Substituting into the display above,

$$\frac{\frac{23(24)}{2} - j}{22} = \frac{23 \cdot 12 - j}{22} = \frac{276 - j}{22}$$

This must equal $\frac{135}{11} = \frac{270}{22}$ which gives $j = 6$.

If $n - 1 = 33$, then $n = 34$. The smallest average occurs if we erase the number 34, leaving the numbers 1 through 33. The average is then

$$\frac{\frac{33(34)}{2}}{33} = \frac{34}{2} = 17$$

which is too high. This value of n and any larger values are impossible. So $j = 6$ is the only possibility.

19. Watson chooses 5 distinct natural numbers and tells Sherlock the product of these numbers. However, it is not enough information for Sherlock to determine whether the sum of the chosen numbers is odd or even. Which of the following could have been the product?

    (A) 1890
    (B) 945
    (C) 180
    (D) 252
    (E) 420

    Correct answers: (E)

    Explanation: $945 = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9$ has only odd factors and so the sum is odd. $1890 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 9$, but since there is only one 2 in the factorization, only one number is even and so the sum is even.

$252 = 2^2 \cdot 3^2 \cdot 7$. The only possibility to get 5 distinct numbers with the given product is $1, 2, 3, 6, 7$. Similarly, $180 = 2^2 \cdot 3^2 \cdot 5$ and also has only way of writing as a product of 5 distinct numbers.

$420 = 2^2 \cdot 3 \cdot 5 \cdot 7$. This has the possibilities $1, 2, 3, 7, 10$ and $1, 3, 4, 5, 7$.

20. A number written in base 8 may start with some number of 0's. We call these *trailing zeroes*. For instance, the number 128 (written in decimal notation) is 200 when written in base 8; so it has 2 trailing zeroes. How many trailing zeroes does the number 100! have when written in base 8?

(A) 28
(B) 20
(C) 44
(D) 26
(E) 32

Correct answers: (E)

Explanation: If 8^k is the largest power of 8 that divides $100!$ then the number of trailing zeroes is k. So we must compute the largest power of 8 that divides $100!$. To do this, we will compute the largest power of 2 that divides $100!$. Each multiple of 2 between 1 and 100 contributes a factor of 2 to $100!$. Every multiple of 4 between 1 and 100 contributes an *additional* factor of 2 to $100!$ and so on. The number of multiples of 2 between 1 and 100 is 50. Dividing this by 2, the number of multiples of 4 between 1 and 100 is 25. The number of multiples of 8 between 1 and 100 is 12. The number of multiples of 16 is 6. The number of multiples of 32 is 3. And the number of multiples of 64 between 1 and 100 is 1. No higher powers of 2 divide any number between 1 and 100. Therefore, the highest power of 2 that divides $100!$ is

$$2^{50+25+12+6+3+1} = 2^{97}.$$

Now, this is equal to $2^{3\cdot 32+1} = 2 \cdot 8^{32}$. So the largest power of 8 that divides $100!$ is 8^{32} and $100!$ has 32 trailing zeroes when written in base 8.