Solving Systems of Linear Equations

This document goes over the substitution and elimination methods for equations with two or more variables, as well as the three types of solutions one will reach when solving these systems.

Method of Substitution

The *method of substitution* starts by solving for one variable in an equation in the system. The resulting equation is then substituted back into the unused equation(s) in the system to solve for the value of all variables.

Example

Solve the given system of equations:

$$\begin{cases} 2x + 3y = 3 \\ 3x - 4y = 13 \end{cases}$$

STEP 1

Number each equation.

$$(1)\ 2x + 3y = 3 \qquad (2)\ 3x - 4y = 13$$

STEP 2

Choose one equation and solve for one variable.

$$(1)\ 2x + 3y = 3$$

$$2x = 3 - 3y$$

$$\boxed{x = \frac{3}{2} - \frac{3}{2}y}$$

STEP 3

Substitute the solved expression in step two for the equivalent variable in (2) and solve for the remaining variable.

$$(2)\ 3\left[\frac{3}{2} - \frac{3}{2}y\right] - 4y = 13$$

$$\frac{9}{2} - \frac{9}{2}y - 4y = 13$$

$$9 - 9y - 8y = 26$$

$$9 - 17y = 26$$

$$-17y = 17$$

$$\boxed{y = -1}$$

STEP 4

Substitute the solution for the variable into either of the original two equations and solve for the other variable.

$$(1)\ 2x + 3[-1] = 3$$

$$2x - 3 = 3$$

$$2x = 6$$

$$\boxed{x = 3}$$

STEP 5

Write your solution as an ordered pair.

$$(3, -1)$$

Method of Elimination

The goal of the *method of elimination* is to solve a system of equations by gradually reducing the number of variables. We do this by lining up the equations and combining them in a way that makes one variable disappear. Usually, this means multiplying one or both equations so that one variable has the same coefficient in both. Once that's true, we can add or subtract the equations to "eliminate" that variable. What's left is an equation with fewer variables, which is easier to solve. By repeating this process, eventually a numerical value is found for one variable. Then, we substitute that value back into earlier equations to uncover the others.

Example

Solve the given system of equations

$$\begin{cases} 2x + y + 2z = 11 \\ 3x + 2y + 2z = 8 \\ x + 4y + 3z = 0 \end{cases}$$

STEP 1
Number each equation.

$$(1)\ 2x + y + 2z = 11$$

$$(2)\ 3x + 2y + 2z = 8$$

$$(3)\ x + 4y + 3z = 0$$

STEP 2
Choose any two equations and eliminate one variable using the elimination method. We will eliminate the y variable by multiplying equation (1) by -2 and adding it to equation (2). The new equation is equation (4).

$$\begin{array}{rr} -2(1) & -4x - 2y - 4z = -22 \\ (2) & 3x + 2y + 2z = 8 \\ \hline (4) & -x - 2z = -14 \end{array}$$

STEP 3
Choose a different pair of equations and eliminate the same variable. We will eliminate the y variable by multiplying equation (2) by -2 and adding it to equation (3). The new equation is equation (5).

$$\begin{array}{rr} -2(2) & -6x - 4y - 4z = -16 \\ (3) & x + 4y + 3z = 0 \\ \hline (5) & -5x - z = -16 \end{array}$$

STEP 4

Equations (4) and (5) create a system of two equations in two variables. Eliminate one of the variables, solve for the remaining variable, then substitute the result back into equation (4) or (5) to solve for the other variable in that system. We will multiply equation (5) by -2 to eliminate the z variable.

$$(4)\ -x-2z=-14$$

$$-2(5)\ 10x+2z=32$$

$$9x=18$$

$$\boxed{x=2}$$

$$(4)\ -[2]-2z=-14$$

$$-2z=-12$$

$$\boxed{z=6}$$

STEP 5

Substitute the solutions for the two variables you have solved for into any one of the original three equations and solve for the third variable.

$$(3)\ [2]+4y+3[6]=0$$

$$2+4y+18=0$$

$$4y=-20$$

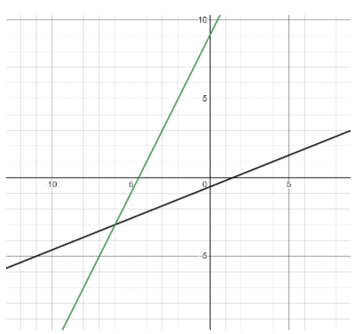
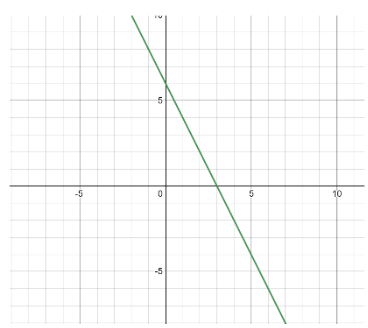
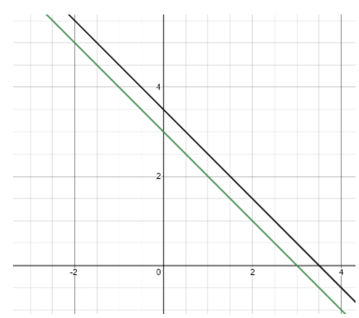
$$\boxed{y=-5}$$

STEP 6

Write your solution as an ordered triple.

$$(2,-5,6)$$

Solution-types for Systems of Linear Equations in Two Variables

Given	$\begin{cases} 2x - 5y = 3 \\ y - 2x = 9 \end{cases}$	$\begin{cases} 4x + 2y = 12 \\ -2x - y = -6 \end{cases}$	$\begin{cases} x + y = 3 \\ 2x + 2y = 7 \end{cases}$
Solve Algebraically	$y - 2x = 9$ $\boldsymbol{y = 2x + 9}$ $2x - 5(2x + 9) = 3$ $2x - 10x - 45 = 3$ $-8x = 48$ $\boldsymbol{x = -6}$ $y - 2(-6) = 9$ $y + 12 = 9$ $\boldsymbol{y = -3}$ $\boldsymbol{(-6, -3)}$	$4x + 2y = 12$ $2(-2x - y = -6)$ $4x + 2y = 12$ $-4x - 2y = -12$ ――――――― $\mathbf{0 = 0}$ Since $0 = 0$ is a **true** statement, these equations will have the same slope and the same y-intercept. They are the same line.	$x + y = 3$ $\boldsymbol{y = 3 - x}$ $2x + 2(3 - x) = 7$ $2x + 6 - 2x = 7$ $\mathbf{6 = 7}$ Since $6 = 7$ is a **false** statement, these equations will have the same slope, but different y-intercepts. They are parallel lines.
Solve Graphically	$\begin{cases} 2x - 5y = 3 \\ y - 2x = 9 \end{cases} \rightarrow \begin{cases} y = \frac{2}{5}x - \frac{3}{5} \\ y = 2x + 9 \end{cases}$ 	$\begin{cases} 4x + 2y = 12 \\ -2x - y = -6 \end{cases} \rightarrow \begin{cases} y = -2x + 6 \\ y = -2x + 6 \end{cases}$ 	$\begin{cases} x + y = 3 \\ 2x + 2y = 7 \end{cases} \rightarrow \begin{cases} y = -x + 3 \\ y = -x + \frac{7}{2} \end{cases}$ 
Then We Say…	Consistent System Independent Equations One Unique Solution	Consistent System Dependent Equations Infinitely Many Solutions	Inconsistent System No Solution