Graphing Rational Functions

This handout goes over graphing rational functions, and how to graph and identify holes and asymptotes.

Overview

There are seven steps that are typically used to graph a rational function.

1. Factor the numerator and denominator
2. Identify the x-intercepts
3. Identify the holes and vertical asymptotes
4. Create a table or number line of critical values to determine intervals where the function is positive or negative
5. Identify horizontal or slant asymptotes
6. Identify the y-intercept
7. Graph all information found in the previous steps

Example

Graph the following rational function:

$$f(x) = \frac{2(x+3)(x^2-3x-4)}{(x+3)(x^2-4)}$$

STEP 1
Given a rational function, factor the numerator and denominator.

$$f(x) = \frac{2(x+3)(x-4)(x+1)}{(x+3)(x-2)(x+2)}$$

STEP 2
Identify the *x*-intercepts.

x-intercepts are found by setting $f(x) = y = 0$.

$$\frac{2\cancel{(x+3)}(x-4)(x+1)}{\cancel{(x+3)}(x-2)(x+2)} = 0$$
$$(x-2)(x+2) \cdot \frac{2(x-4)(x+1)}{(x-2)(x+2)} = 0 \cdot (x-2)(x+2)$$
$$2(x-4)(x+1) = 0$$

Using the zero product property, we can set each factor equal to zero.

$$x - 4 = 0 \rightarrow x = 4$$

$$x + 1 = 0 \rightarrow x = -1$$

STEP 3
Identify the holes and vertical asymptotes.

A vertical asymptote is an invisible line the graph will never cross and is written as the equation of a vertical line, $x = a$. The vertical asymptotes of a rational function are directly related to its domain restrictions, or in other words, vertical asymptotes occur when our denominator equals zero.

Holes are also related to domain restrictions and occur when we have identical factors on our numerator and denominator that create division by zero but also cancel out.

$$f(x) = \frac{2(x+3)(x-4)(x+1)}{(x+3)(x-2)(x+2)}$$

So, we have a hole when

$$x + 3 = 0$$
$$x = -3$$

And we can find the *y*-coordinate of the hole by finding $f(-3)$

$$f(x) = \frac{2\cancel{(x+3)}(x-4)(x+1)}{\cancel{(x+3)}(x-2)(x+2)}$$
$$f(-3) = \frac{2(-3-4)(-3+1)}{(-3-2)(-3+2)}$$
$$= \frac{28}{5}$$
$$= 5.6$$

Then our hole is located at

$$(-3, 5.6)$$

And our vertical asymptotes are located at

$$x - 2 = 0 \rightarrow x = 2$$

$$x + 2 = 0 \rightarrow x = -2$$

STEP 4

Create a number line or table using the critical values (x-intercepts and vertical asymptotes) identified above as interval boundaries to determine the sign of each interval. Note that we aren't looking at the factor $x + 3$ from the numerator and denominator since they cancel out.

	$(-\infty, -2)$	$(-2, -1)$	$(-1, 2)$	$(2, 4)$	$(4, \infty)$
$x - 4$	$-$	$-$	$-$	$-$	$+$
$x + 1$	$-$	$-$	$+$	$+$	$+$
$x - 2$	$-$	$-$	$-$	$+$	$+$
$x + 2$	$-$	$+$	$+$	$+$	$+$
$f(x)$	$+$	$-$	$+$	$-$	$+$

STEP 5

Identify horizontal or slant asymptotes.

Horizontal and slant asymptotes are invisible lines the graph approaches as the x-values approach $\pm\infty$. Unlike vertical asymptotes, the graph can cross through horizontal and slant asymptotes. Horizontal asymptotes are of the form $y = a$ and slant asymptotes are of the form $y = mx + b$.

Horizontal and slant asymptotes are found by comparing the degree of the numerator to the degree of the denominator.

$$\text{Given } f(x) = \frac{ax^n + \cdots}{bx^m + \cdots}$$

If…

$$n < m, \qquad y = 0$$

$$n = m, \qquad y = \frac{a}{b}$$

$$n = m + 1, \qquad \text{slant, use polynomial long divison}$$

$$\text{otherwise,} \qquad \text{none}$$

In this case, $n = 3 = m$ so our horizontal asymptote is

$$y = \frac{2}{1} = 2$$

STEP 6
Identify *y*-intercept.

y-intercepts are found by setting $x = 0$.

$$f(0) = \frac{2(0+3)(0-4)(0+1)}{(0+3)(0-2)(0+2)}$$
$$= -\frac{24}{-12}$$
$$= 2$$

STEP 7

Graph.

1. Graph the asymptotes.
2. Graph the *x*- and *y*-intercepts.
3. Graph any holes.
4. Graph the curves, using the asymptotes and the intervals in step 4 as guides.