Partial Fraction Decomposition

UVU Math Lab

Using Systems of Equations:

Find the partial fraction decomposition for the following rational expression: \( \frac{2}{x^5+x^3-x^4-x^2} \)

**STEP 1:** Factor the denominator and set expression equal to the form of partial fraction decomposition with the unknown constants \((A, B, C,...)\) in the numerators of the decomposition.

\[
\frac{2}{x^2(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}
\]

**STEP 2:** Multiply both sides by the LCD to eliminate denominators and multiply through.

\[
x^2(x-1)(x^2+1) \cdot \frac{2}{x^2(x-1)(x^2+1)} = \frac{A}{x} \cdot x^2(x-1)(x^2+1) + \frac{B}{x^2} \cdot x^2(x-1)(x^2+1) + \frac{C}{x-1} \cdot x^2(x-1)(x^2+1) + \frac{Dx+E}{x^2+1} \cdot x^2(x-1)(x^2+1)
\]

\[
2 = Ax(x-1)(x^2+1) + B(x-1)(x^2+1) + Cx^2(x^2+1) + (Dx+E)x^2(x-1)
\]

\[
2 = Ax^4 + Ax^2 - Ax^3 + Bx^3 + Bx - Bx^2 - B + Cx^4 + Cx^2 + Dx^4 - Dx^3 + Ex^2 - Ex^2
\]

**STEP 3:** Group like terms and write both sides of the equation in descending powers of \( x \).

\[
0x^4 + 0x^3 + 0x^2 + 0x^1 + 2x^0 = x^4(A + C + D) + x^3(-A + B - D + E) + x^2(A - B + C - E) + x(-A + B) - B
\]

**STEP 4:** Equate the coefficients of like powers and solve the resulting system of equations.

(4) \( x^4: 0 = A + C + D \)
(3) \( x^3: 0 = -A + B - D + E \)
(2) \( x^2: 0 = A - B + C - E \)
(1) \( x^1: 0 = -A + B \)
(0) \( x^0: 2 = -B \)

(0) \( 2 = -B \) \rightarrow \( B = -2 \)
(1) \( 0 = -A + (-2) \) \rightarrow \( A = -2 \)
(2) \( 0 = (-2) - (-2) + C - E \)
(3) \( 0 = (-2) - (-2) + C - E \)
(4) \( 0 = (-2) + C + D \)
(5) \( 0 = \frac{C}{D} \)
(6) \( 0 = 2 + 2C \)

\( C = 1 \)
\( D = 1 \)

**STEP 5:** Substitute the values found for the unknown constants back into the decomposition.

\[
\frac{2}{x^2(x-1)(x^2+1)} = \frac{-2}{x} + \frac{2}{x^2} + \frac{1}{x-1} + \frac{x+1}{x^2+1}
\]

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Partial Fraction Decomposition

UVU Math Lab

Using the Zero-Out Method:  
(The Short Cut!)

Find the partial fraction decomposition for the following rational expression: \( \frac{2}{x^5+x^3-x^4-x^2} \)

<table>
<thead>
<tr>
<th>STEP 1: Factor the denominator and set expression equal to the form of partial fraction decomposition with the unknown constants (A, B, C,...) in the numerators of the decomposition.</th>
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<tbody>
<tr>
<td>( \frac{2}{x^2(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1} )</td>
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<th>STEP 2: Multiply both sides by the LCD to eliminate denominators.</th>
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<td>( x^2(x-1)(x^2+1) \cdot \frac{2}{x^2(x-1)(x^2+1)} = A \cdot x^2(x-1)(x^2+1) + B \cdot x^2(x-1)(x^2+1) + C \cdot x^2(x-1)(x^2+1) + (Dx+E) \cdot x^2(x-1)(x^2+1) )</td>
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<td>( 2 = Ax(x-1)(x^2+1) + B(x-1)(x^2+1) + Cx^2(x^2+1) + (Dx+E)x^2(x-1) )</td>
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<tr>
<th>STEP 3: Choose a value for ( x ) that will result in zero factors and solve for the remaining letter.</th>
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<tr>
<td>Let ( x = 0 ):</td>
</tr>
<tr>
<td>( 2 = 0 + B(-1)(1) + 0 + 0 )</td>
</tr>
<tr>
<td>( 2 = -B )</td>
</tr>
<tr>
<td>( B = -2 )</td>
</tr>
<tr>
<td>Let ( x = 1 ):</td>
</tr>
<tr>
<td>( 2 = 0 + 0 + C(1)(2) + 0 )</td>
</tr>
<tr>
<td>( 2 = 2C )</td>
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<td>( C = 1 )</td>
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<th>STEP 4: Substitute solved values into equation and then begin with STEP 3 on opposite side to solve any remaining values using systems of equations.</th>
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<tr>
<td>( 2 = Ax(x-1)(x^2+1) - 2(x-1)(x^2+1) + 1x^2(x^2+1) + (Dx+E)x^2(x-1) )</td>
</tr>
<tr>
<td>( 2 = Ax^4 + Ax^3 - Ax - 2x^3 - 2x + 2x^2 + 2 + 1x^4 + 1x^2 + Dx^4 - Dx^3 + Ex^3 - Ex^2 )</td>
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<tr>
<td>( 2 = x^4(A + 1 + D) + x^3(-A - 2 - D - E) + x^2(A + 2 + 1 - E) + x(-A - 2) + 2 )</td>
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\( x^4: -1 = A + D \)  
\( x^3: 2 = -A - D + E \)  
\( x^2: -3 = A - E \)  
\( x^1: 2 = -A \)  

\( 1 \) \( A = -2 \)  
\( 2 \) \( -3 = -2 - E \quad \rightarrow E = 1 \)  
\( 4 \) \( -1 = -2 + D \quad \rightarrow D = 1 \)  

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<th>STEP 5: Substitute the values found for the unknown constants back into the decomposition.</th>
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<td>( \frac{2}{x^2(x-1)(x^2+1)} = \frac{-2}{x} - \frac{2}{x^2} + \frac{1}{x-1} + \frac{x+1}{x^2+1} )</td>
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The Four Types of Partial Fraction Decomposition:

Previously, we learned that in order to find the sum of rational expressions we had to find a common denominator.

\[
\frac{x}{x+1} + \frac{2}{x-1} - \frac{3}{x} = \frac{x + (x-1)(x+1)}{x(x-1)} - \frac{3(x+1)(x-1)}{x(x-1)(x+1)}
\]

With partial fraction decomposition, we reverse that process and split the rational expression into partial fractions, expressions with irreducible denominators.

Given the rational expression \( \frac{P(x)}{Q(x)} \), we factor the denominator, \( Q(x) \) and then set the expression equal to the partial fraction decomposition according to the following forms.

**Type I:**
The denominator is the product of distinct, non-repeating linear factors, such as \((x - a)\):

\[
\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \ldots
\]

**Type II:**
The denominator is the product of repeating linear factors, such as \((x - a)^n\):

\[
\frac{P(x)}{Q(x)} = \frac{C_1}{x-a} + \frac{C_2}{(x-a)^2} + \ldots + \frac{C_n}{(x-a)^n}
\]

**Type III:**
The denominator is the product of distinct irreducible quadratic factors (cannot be factored further) of the form \((ax^2 + bx + c)\):

\[
\frac{P(x)}{Q(x)} = \frac{Ax+B}{x^2+a} + \frac{Cx+D}{x^2+b} + \frac{Dx+E}{x^2+c} + \ldots
\]

**Type IV:**
The denominator is the product of repeating irreducible quadratic factors, such as \((ax^2 + bx + c)^n\):

\[
\frac{P(x)}{Q(x)} = \frac{C_1 x + C_2}{x^2+a} + \frac{C_3 x + C_4}{(x^2+a)^2} + \ldots + \frac{C_{m-1} x + C_m}{(x^2+a)^n}
\]

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