Partial Fraction Decomposition

UVU Math Lab

Using Systems of Equations:

Find the partial fraction decomposition for the following rational expression: $\frac{2}{x^5+x^3-x^4-x^2}$

STEP 1: Factor the denominator and set expression equal to the form of partial fraction decomposition with the unknown constants (A, B, C,…) in the numerators of the decomposition.

$$\frac{2}{x^2(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}$$

STEP 2: Multiply both sides by the LCD to eliminate denominators and multiply through.

$$\cancel{x^2(x-1)(x^2+1)} \cdot \frac{2}{\cancel{x^2(x-1)(x^2+1)}} = \frac{A}{\cancel{x}} \cdot \cancel{x^2}(x-1)(x^2+1) + \frac{B}{\cancel{x^2}} \cdot \cancel{x^2}(x-1)(x^2+1) + \frac{C}{\cancel{x-1}} \cdot x^2\cancel{(x-1)}(x^2+1) + \frac{Dx+E}{\cancel{x^2+1}} \cdot x^2(x-1)\cancel{(x^2+1)}$$

$$2 = Ax(x-1)(x^2+1) + B(x-1)(x^2+1) + Cx^2(x^2+1) + (Dx+E)x^2(x-1)$$

$$2 = Ax^4 + Ax^2 - Ax^3 - Ax + Bx^3 + Bx - Bx^2 - B + Cx^4 + Cx^2 + Dx^4 - Dx^3 + Ex^3 - Ex^2$$

STEP 3: Group like terms and write both sides of the equation in descending powers of x.

$$0x^4 + 0x^3 + 0x^2 + 0x^1 + 2x^0$$
$$= x^4(A+C+D) + x^3(-A+B-D+E) + x^2(A-B+C-E) + x(-A+B) - B$$

STEP 4: Equate the coefficients of like powers and solve the resulting system of equations.

(4) x^4: $0 = A + C + D$ (3) x^3: $0 = -A + B - D + E$ (2) x^2: $0 = A - B + C - E$ (1) x^1: $0 = -A + B$ (0) x^0: $2 = -B$	(0) $2 = -B$ $\rightarrow$ $\mathbf{B = -2}$ (1) $0 = -A + (-2)$ $\rightarrow$ $\mathbf{A = -2}$ (3) $0 = -(-2) + (-2) - D + E$ (2) $\underline{0 = (-2) - (-2) + C - E}$ (5) $0 = C - D$	(4) $0 = (-2) + C + D$ (5) $\underline{0 = \quad C - D}$ (6) $0 = \quad 2 + 2C$ $\rightarrow$ $\mathbf{C = 1}$ (2) $0 = -2 + 2 + 1 - E$ $\rightarrow$ $\mathbf{E = 1}$ (3) $0 = 2 - 2 - D + 1$ $\rightarrow$ $\mathbf{D = 1}$

STEP 5: Substitute the values found for the unknown constants back into the decomposition.

$$\frac{2}{x^2(x-1)(x^2+1)} = -\frac{2}{x} - \frac{2}{x^2} + \frac{1}{x-1} + \frac{x+1}{x^2+1}$$

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Using the Zero-Out Method: (The Short Cut!)

Find the partial fraction decomposition for the following rational expression: $\frac{2}{x^5+x^3-x^4-x^2}$

STEP 1: Factor the denominator and set expression equal to the form of partial fraction decomposition with the unknown constants (A, B, C,…) in the numerators of the decomposition.

$$\frac{2}{x^2(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}$$

STEP 2: Multiply both sides by the LCD to eliminate denominators.

$$\cancel{x^2(x-1)(x^2+1)}\cdot\frac{2}{\cancel{x^2(x-1)(x^2+1)}} = \frac{A}{\cancel{x}}\cdot x^{\cancel{2}}(x-1)(x^2+1) + \frac{B}{\cancel{x^2}}\cdot\cancel{x^2}(x-1)(x^2+1) + \frac{C}{\cancel{x-1}}\cdot x^2\cancel{(x-1)}(x^2+1) + \frac{Dx+E}{\cancel{x^2+1}}\cdot x^2(x-1)\cancel{(x^2+1)}$$

$$2 = Ax(x-1)(x^2+1) + B(x-1)(x^2+1) + Cx^2(x^2+1) + (Dx+E)x^2(x-1)$$

STEP 3: Choose a value for x that will result in zero factors and solve for the remaining letter.

Let $x = 0$:

$2 = 0 + B(-1)(1) + 0 + 0$

$2 = -B$

$\boldsymbol{B = -2}$

Let $x = 1$:

$2 = 0 + 0 + C(1)(2) + 0$

$2 = 2C$

$\boldsymbol{C = 1}$

STEP 4: Substitute solved values into equation and then begin with STEP 3 on opposite side to solve any remaining values using systems of equations.

$$2 = Ax(x-1)(x^2+1) - 2(x-1)(x^2+1) + 1x^2(x^2+1) + (Dx+E)x^2(x-1)$$

$$2 = Ax^4 + Ax^2 - Ax^3 - Ax - 2x^3 - 2x + 2x^2 + 2 + 1x^4 + 1x^2 + Dx^4 - Dx^3 + Ex^3 - Ex^2$$

$$2 = x^4(A+1+D) + x^3(-A-2-D+E) + x^2(A+2+1-E) + x(-A-2) + 2$$

(4) $x^4: -1 = A + D$

(3) $x^3: 2 = -A - D + E$

(2) $x^2: -3 = A - E$

(1) $x^1: 2 = -A$

(1) $\boldsymbol{A = -2}$

(2) $-3 = -2 - E \quad \rightarrow \boldsymbol{E = 1}$

(4) $-1 = -2 + D \quad \rightarrow \boldsymbol{D = 1}$

STEP 5: Substitute the values found for the unknown constants back into the decomposition.

$$\frac{2}{x^2(x-1)(x^2+1)} = -\frac{2}{x} - \frac{2}{x^2} + \frac{1}{x-1} + \frac{x+1}{x^2+1}$$

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The Four Types of Partial Fraction Decomposition:

Previously, we learned that in order to find the sum of rational expressions we had to find a common denominator.

$$\frac{x}{x+1}+\frac{2}{x-1}-\frac{3}{x}$$

$$=\frac{x}{x+1}\cdot\frac{x(x-1)}{x(x-1)}+\frac{2}{x-1}\cdot\frac{x(x+1)}{x(x+1)}-\frac{3}{x}\cdot\frac{(x+1)(x-1)}{(x-1)(x+1)}$$

$$=\frac{x^3-2x^2+2x+3}{x(x^2-1)}$$

With partial fraction decomposition, we reverse that process and split the rational expression into partial fractions, expressions with irreducible denominators.

Given the rational expression $\frac{P(x)}{Q(x)}$, we factor the denominator, $Q(x)$ and then set the expression equal to the partial fraction decomposition according to the following forms.

Type I:
The denominator is the product of distinct, non-repeating linear factors, such as $(x-a)$:

$$\frac{P(x)}{Q(x)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\cdots$$

Type II:
The denominator is the product of repeating linear factors, such as $(x-a)^n$:

$$\frac{P(x)}{Q(x)}=\frac{C_1}{x-a}+\frac{C_2}{(x-a)^2}+\cdots+\frac{C_n}{(x-a)^n}$$

Type III:
The denominator is the product of distinct irreducible quadratic factors (cannot be factored further) of the form (ax^2+bx+c):

$$\frac{P(x)}{Q(x)}=\frac{Ax+B}{x^2+a}+\frac{Cx+D}{x^2+b}+\frac{Dx+E}{x^2+c}+\cdots$$

Type IV:
The denominator is the product of repeating irreducible quadratic factors, such as $(ax^2+bx+c)^n$:

$$\frac{P(x)}{Q(x)}=\frac{C_1x+C_2}{x^2+a}+\frac{C_3x+C_4}{(x^2+a)^2}+\cdots+\frac{C_{m-1}x+C_m}{(x^2+a)^n}$$