Partial Fraction Decomposition (PFD)

This handout will introduce the key terms related to Partial Fraction Decomposition (PFD), explain the rules for applying it, and walk through examples using all three main cases.

Key Terms

- *Term* – A set of coefficients and/or variables separated by a plus (+) or minus (-) sign.
  $27x^2 + 5x - 7$ has 3 terms: $27x^2$, $5x$, and -7.

- *Factor* – A term or sum of terms, the latter often contained in parenthesis, that create a product when multiplied by another factor.
  $3(x^2 + 4)(x - 5)$ has 3 factors: 3, $(x^2 + 4)$, and $(x - 5)$.

- *Product* – The answer to a multiplication problem.
  The equation $2 \times 5 = 10$ contains factors 2 and 5, resulting in a product of 10.

- *Reducible Factor* – A factor that can be factored further than currently expressed.
  The expression $(x^2 - 4)(x + 2)$ contains factors $(x^2 - 4)$ and $(x + 2)$. $(x^2 - 4)$ is a reducible factor, $(x^2 - 4) = (x + 2)(x - 2)$.

- *Irreducible Quadratic Factor* – A factor in the form of $ax^2 + bx - c$ that cannot be reduced using rational or real numbers.
  While the expression $(x^2 + x + 4)(x^2 - 4)$ has 2 quadratic factors $(x^2 + x + 4)$ cannot be factored, making it an irreducible quadratic factor.

- *Repeated Factor* – A factor that has a multiplicity other than one.
  The expression $x^3(x + 2)^2(x - 2)$ has the factor x repeated 3 times and the factor $(x + 2)$ repeated twice.

PFD Cases

Partial Fraction Decomposition has 3 possible outcomes:

Case 1

When the factor in the denominator is a linear-irreducible factor, it will be re-written as

$$\frac{A}{(ax - r)}$$

Case 2

When the factor is a repeated factor, it will be re-written as

$$\frac{A_1}{(ax - r)} + \frac{A_2}{(ax - r)^2} + \cdots + \frac{A_n}{(ax - r)^n}$$

Case 3

When the factor is an irreducible quadratic, it will be re-written as

$$\frac{Ax + B}{ax^2 + bx + c}$$

Partial Fraction Decomposition Case 1

Setup
Begin with the fraction expression:

$$\frac{x+2}{(x+1)(x-1)}$$

Step 1
The denominator has 2 linear factors: $(x+1)$ and $(x-1)$. This means that the expression will be rewritten like this.

$$\frac{A}{(x+1)}+\frac{B}{(x-1)}$$

Step 2
Set the expression equal to the original expression.

$$\frac{A}{(x+1)}+\frac{B}{(x-1)}=\frac{x+2}{(x+1)(x-1)}$$

Step 3
Multiply both sides by the denominator of the right side of the equation.

$$\frac{A}{(x+1)}\big((x+1)(x-1)\big)+\frac{B}{(x-1)}\big((x+1)(x+1)\big)=x+2$$

Step 4
Cancel out common terms. On the A term, $(x+1)$ cancels out. On the B term $(x-1)$ cancels out.

$$A(x-1)+B(x+1)=x+2$$

Step 5
Distribute A and B, then group the terms based on whether or not they have an x term.

$$Ax-A+Bx+B=x+2 \quad \rightarrow \quad Ax+Bx-A+B=x+2$$

$$(A+B)x+(-A+B)=x+2$$

Step 6
Solve the resulting system of linear equations, where $(A+B)x=x$ and $(-A+B)=2$.

$$A+B=1$$

$$-A+B=2$$

$$2B=3,\ \boxed{B=\tfrac{3}{2}}$$

$$A+\tfrac{3}{2}=1,\ \boxed{A=-\tfrac{1}{2}}$$

Step 7
Plug in A and B for the finished partial fraction decomposition:

$$\frac{-\frac{1}{2}}{(x+1)}+\frac{\frac{3}{2}}{(x-1)}$$

Partial Fraction Decomposition Case 2

Setup
Begin with the fraction expression:

$$\frac{x+2}{(x+4)^2}$$

Step 1
Notice that $(x+4)^2$ is a repeated linear factor. Following the steps previously stated for this case; split the fraction and have each successive fraction increase the denominator degree until the original degree is reached.

$$\frac{A}{(x+4)}+\frac{B}{(x+4)^2}$$

Step 2
Set this equal to the original fraction.

$$\frac{A}{(x+4)}+\frac{B}{(x+4)^2}=\frac{x+2}{(x+4)^2}$$

Step 3
Multiply both sides by the denominator on the right side of the equation.

$$\frac{A}{(x+4)}(x+4)^2+\frac{B}{(x+4)^2}(x+4)^2=\frac{x+2}{(x+4)^2}(x+4)^2$$

Step 4
Simplify and distribute factors.

$$A(x+4)+B=x+2$$

$$Ax+4A+B=x+2$$

Step 5
Solve for A and B as previously done, grouping terms into a system of equations and solving.

$$Ax=x, \boxed{A=1}$$

$$4A+B=2$$

$$4(1)+B=2$$

$$\boxed{B=\frac{1}{2}}$$

Step 6
Plug in A and B for the final partial fraction decomposition:

$$\frac{1}{(x+4)}+\frac{\frac{1}{2}}{(x+4)^2}$$

Partial Fraction Decomposition Case 3

Setup
Begin with the fraction expression:

$$\frac{x+2}{x^2+4}$$

Step 1
Notice that $x^2 + 4$ does not have any real zeros. That means the polynomial will be rewritten like this:

$$\frac{Ax+B}{x^2+4}$$

Step 2
Set the rewritten version equal to the original fraction.

$$\frac{Ax+B}{x^2+4} = \frac{x+2}{x^2+4}$$

Step 3
Multiply both sides by the denominator on the right side of the equation.

$$\frac{A}{x^2+4}(x^2+4) = \frac{x+2}{x^2+4}(x^2+4)$$

Step 4
Simplify until the fractions are eliminated.

$$Ax + B = x + 2$$

Step 5
The equation can now be set to a system of linear equations.

$$A = 1 \quad B = 2$$

Step 6
Plug in A and B for the final partial fraction decomposition:

$$\frac{x+2}{x^2+4}$$