# Proofs Crash Course

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*Math Lab*

# Today’s Topics

- Why are Proofs so Hard?
- Proof by Deduction
- Proof by Contrapositive
- Proof by Contradiction
- Proof by Induction

# Why are Proofs so Hard?

"If it is a miracle, any sort of evidence will answer, but if it is a fact, proof is necessary"

-Mark Twain

# Why are Proofs so Hard?

- Proofs are very different from the math problems that you're used to in High School.
- Proofs are problems that require a whole different kind of thinking.
- Most proofs will not give you all of the information you need to complete them.

# Before we go further…

- Understanding the purpose of proofs is fundamental to understanding how to solve them.

- Doing proofs is like making a map
  - The goal is to get from point A to B using paths, roads, and highways.
  - Proofs show us how two statements logically connect to each other through theorems, definitions, and laws.

# Proof by Deduction

"The two operations of our understanding, intuition and deduction, on which alone we have said we must rely in the acquisition of knowledge."

-Rene Descartes

# Proof by Deduction

- This is the most basic proof technique.
- By using laws, definitions, and theorems you can get from A to B by starting at A and progressively moving towards B.
- You start by assuming the conditional (the “if” part) and showing the logical flow to the conclusion (the “then” part).

# Deductive Proof Example

Suppose you know the following:

if A then B

if B then C

if C then D

Show that if A then D.

# Deductive Proof Example

Remember that deductive proofs start at the beginning and proceed towards the conclusion

<u>**Proof:**</u> Assume A is true. Therefore B must be true. Since B is true, C is true. Because C is true, D is true.

Hence, D. ■

# Deductive Proof Analysis

- Notice that the path taken to get from A to D was very direct and linear.
- We started by assuming that A was true.
- Then we used the given “laws” to show that D was true.

# Deductive Proof Example

Prove the following statement:

If Jerry is a jerk, Jerry won't get a family.

Note: Many of you likely can prove this using some form of intuition. However, in order to definitively prove something, there need to be some agreed upon guidelines.

# Deductive Proof Example

If Jerry is a jerk, Jerry won't get a family.

Let's also suppose that we have some guidelines:

- If somebody doesn't date, they won't get married.
- If you don't get married, you won't get a family.
- Girls don't date jerks.

# Deductive Proof Example

How would you prove that Jerry won't get a family if he's a jerk?

Given:

1. If somebody doesn't date, they won't get married.
2. If you don't get married, you won't get a family.
3. Girls don't date jerks.
4. Jerry is a jerk. (Our assumption)

Conclusion:

Jerry won't get a family.

# Deductive Proof Solution

Proof:

Suppose that Jerry is a jerk. We therefore know that girls don't date him. Therefore he will never get married. Hence, he won't get a family. ■

# Deductive Proof Example

Prove the following:

if f(x) is even, then f(x) is not one-to-one

# Deductive Proof Example

if f(x) is even, then f(-x) is not one-to-one.

- How is this problem different from our previous ones?
  - What does it mean for f(x) to be even?
  - What does it mean for f(-x) to be one-to-one?
- This is a main reason why proofs are so hard; they don't give you all of the information you need to solve the problem.

# Deductive Proof Example

Remember the following:

f(x) is even if f(-x) = f(x) for all values of x.

f(x) is one-to-one if for all x, f(x) is unique.

How would you solve the problem?

if f(x) is even, then f(x) is not one-to-one.

# Deductive Proof Solution

<u>**Proof:**</u>

Suppose that f(x) is even. This implies that f(x) = f(-x) for all x. Therefore there exists an a and b such that f(a) = f(b). Therefore f(x) is not one-to-one. ∎

Note: Notice how we used a lot of words instead of math symbols? They are still present, but the main way of communicating with math is through using English. Who knew?

# A Word of Caution

A common pitfall students fall in while solving proofs is by assuming that the **converse** of a statement is true.

BE EXTREMELY CAREFUL WHEN DOING THIS!

If it is raining outside, then the sidewalk is wet.

If the sidewalk is wet, then it is raining outside.

# When the Converse is True

- If the door is locked, I can't open it
- If the door is unlocked, I can open it

- Statements whose converse is guaranteed to be true have an "if and only if" clause.

- If and only if the door is locked, I can't open it.
- I can't open the door if and only if it is locked.

# The Biconditional

- The clause "if and only if" means that the statement is true read both ways.
- A if and only if B is the same as saying:
  - If A, then B
  - If B, then A

- Proving statements with an "if and only if" clause require us to show they are true in both directions.

# Deductive Proof Example

Prove the following:

x is even if and only if x + 1 is odd

Note:

An even number y can be represented by $y = 2k$ for some integer k.

Similarly, an odd number z can be represented by $z = 2j + 1$ for some integer j.

# Deductive Proof Solution

**Proof:**

Suppose that x is even. This means that there exists an integer k such that $x = 2k$. Therefore, $x + 1 = 2k + 1$. Since k is an integer, $x + 1$ must be odd.

Now suppose that $x + 1$ is odd. This means that there exists an integer j such that $x + 1 = 2j + 1$, or in other words, $x = 2j$. Since j is an integer, x must therefore be even. ■

# Deductive Proof Challenge

Prove the following:

$x^2 - 2x + 2 \geq 0$ for all x

# Proof by Contrapositive

If the dog is dead, he smells.

If he doesn’t smell, he’s not dead.

# Proof by Contrapositive

- Remember that the **Converse** is not always true.

- The **Contrapositive** is similar to the converse, but is always true.
  - If A then B ≡ If not B then not A
  - If it is raining, the sidewalk is wet ≡ If the sidewalk is **not** wet then it is **not** raining.

# Proof by Contrapositive

- This proof technique makes a lot of proofs so much easier.
- Sometimes the direct route is just too difficult to deduce.
- Example: If $3x + 7$ is odd, then $x$ is even.

# Contrapositive Example

Prove the following:

If $3x + 7$ is even, then $x$ is odd.

# Contrapositive Solution

**Proof:**

Suppose that x is even. This implies that $x = 2k$ for some integer k. Thus, $3x + 7 = 3(2k) + 7 = 6k + 7 = 6k + 6 + 1 = 2(3k + 3) + 1$. And since $3k + 3$ is an integer, then $3x + 7$ is even. ■

Note: this may seem odd, assuming the opposite of the conclusion and proving the opposite of the condition, but this is perfectly legitimate.

# Contrapositive Example

Prove the following:

If $x^2$ is even, then x is even.

# Contrapositive Solution

**Proof:**

Suppose that $x$ is odd. This implies that $x = 2k + 1$ for some integer $k$. Thus,

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since $2k^2 + 2k$ is an integer, then $x^2$ is odd. ■

# Contrapositive Challenge

Prove the following:

If $3n^5$ is an odd integer, then n is an odd integer.

# Proof by Contradiction

"Contradictions do not exist. Whenever you think you are facing a contradiction, check your premises. You will find that one of them is wrong."

-Ayn Rand

# Proof by Contradiction

- If you could prove that a statement could **never** be false, then it **must** be true.
- For example, Let's assume that somebody in this room robbed the bookstore about 2 minutes ago.
- Can we prove that they didn't commit the crime?

# Proof by Contradiction

- Well, you were all here during the last two minutes.  Therefore you couldn't have committed the crime.  Therefore you are innocent.

- That is the basic structure of a proof by contradiction:
  - Assume the conclusion is false
  - Find a contradiction

# Contradictive Proof Example

Prove the following:

No odd integer can be expressed as the sum of three even integers.

Note: there is no “if” or “then” clause, and the statement sounds negative. This is usually a hint that proof by contradiction is the method of choice.

# Contradictive Proof Solution

**Proof:**

Assume to the contrary that an odd integer $x$ can be expressed as the sum of three even integers, $a$, $b$, and $c$. This implies that there exist integers $i$, $j$, and $k$ such that $a = 2i$, $b = 2j$, $c = 2k$. Thus:

$$x = a + b + c = 2i + 2j + 2k = 2(i + j + k)$$

Since $i + j + k$ is an integer, then $x$ is even, a contradiction. ∎

# Contradictive Proof Example

Prove the following:

There is no largest positive integer.

# Contradictive Proof Solution

Proof:

Assume to the contrary that there exists a largest positive integer, notated by k. However, k+1 is an integer and k+1 > k. Therefore k+1 is a larger integer, a contradiction. ■

# Contradictive Proof Challenge

Prove the following:

There do not exist integers x and y such that $9 = 4x + 2y$.

# Proof by Induction

"The only hope [of science] … is in genuine induction"

-Sir Francis Bacon

# Proof by Induction

- This proof technique is used when you want to show that something works for several cases.
- These cases are typically denoted by “n,” representing positive integers.

# Proof by Induction

- Proof by Induction can prove statements such as:

  $1 + 2 + \ldots + n = n(n + 1)/2$, for all n

# Proof by Induction

- There is a very systematic way to prove this:
  1. Prove that it works for a base case (n = 1)
  2. Assume it works for n = k
  3. Show that is works for n = k + 1
- Think of this as a row of dominoes.
  1. Knock over the first domino
  2. Assume that a random one will get knocked over
  3. Show that the random one will hit the next one.
- Thus all of the dominoes get knocked down.

# Inductive Proof Example

Prove the following:

$1 + 2 + \ldots + n = n(n + 1)/2$, for all n

# Inductive Proof Solution

<u>**Proof:**</u>

Let $n = 1$. Since $1 = 1(2)/2 = 1(1 + 1)/2$, then the statement holds for a base case.

Now assume that it holds for $n = k$, or in other words:

$$1 + 2 + \ldots + k = k(k + 1)/2$$

Hence,

$$1 + 2 + \ldots + k + (k + 1) = k(k + 1)/2 + (k + 1)$$

$$= (k + 1)(k/2 + 1) = (k + 1)(k + 2)/2 = (k + 1)((k+1) + 1)/2$$

Therefore it holds for $n = k + 1$, and the statement is true by induction.■

# Inductive Proof Example

Prove the following:

$2^n > n$ for all nonnegative integers

# Inductive Proof Solution

**Proof:**

Let $n = 0$. Thus $2^0 = 1 > 0$, and the statement holds for $n = 0$.

Now assume that $2^k > k$.

Hence,

$$2^{k+1} = (2)2^k > 2k = k + k \geq k + 1$$

Thus the statement holds for $n = k + 1$. Therefore the statement is true by induction. ∎

# Inductive Proof Challenge

Prove the following:

$2 * 6 * 10 * . . . * (4n – 2) = (2n)!/n!$, for all n

# Conclusion

"Don't cry because it's over. Smile because it happened."

-Dr. Seuss

# A Few Words of Wisdom

- You are now very well-equipped to handle almost any proof that comes your way.
- Even so, you will likely get stuck in the future.

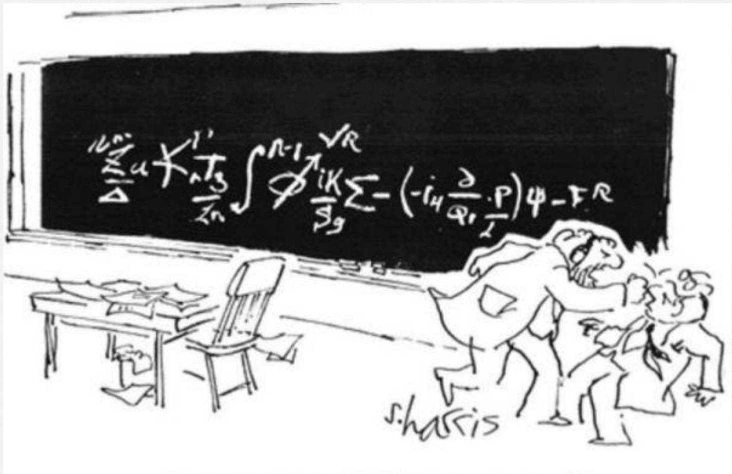
# A Few Words of Wisdom

Remember these tips when you get stuck:

1. Start writing something!
   - The epiphanies come when you start messing around with ideas.
2. Check to see if you've overlooked a theorem
   - So much suffering and head-banging is prevented by simply re-skimming the chapter.
3. Try a fresh approach
   - Sometimes our first ideas just aren't the right ones.

# Conclusion

Now go show your professors that they can't intimidate you with proofs anymore!




"You want proof? I'll give you proof!"

Thanks for coming!